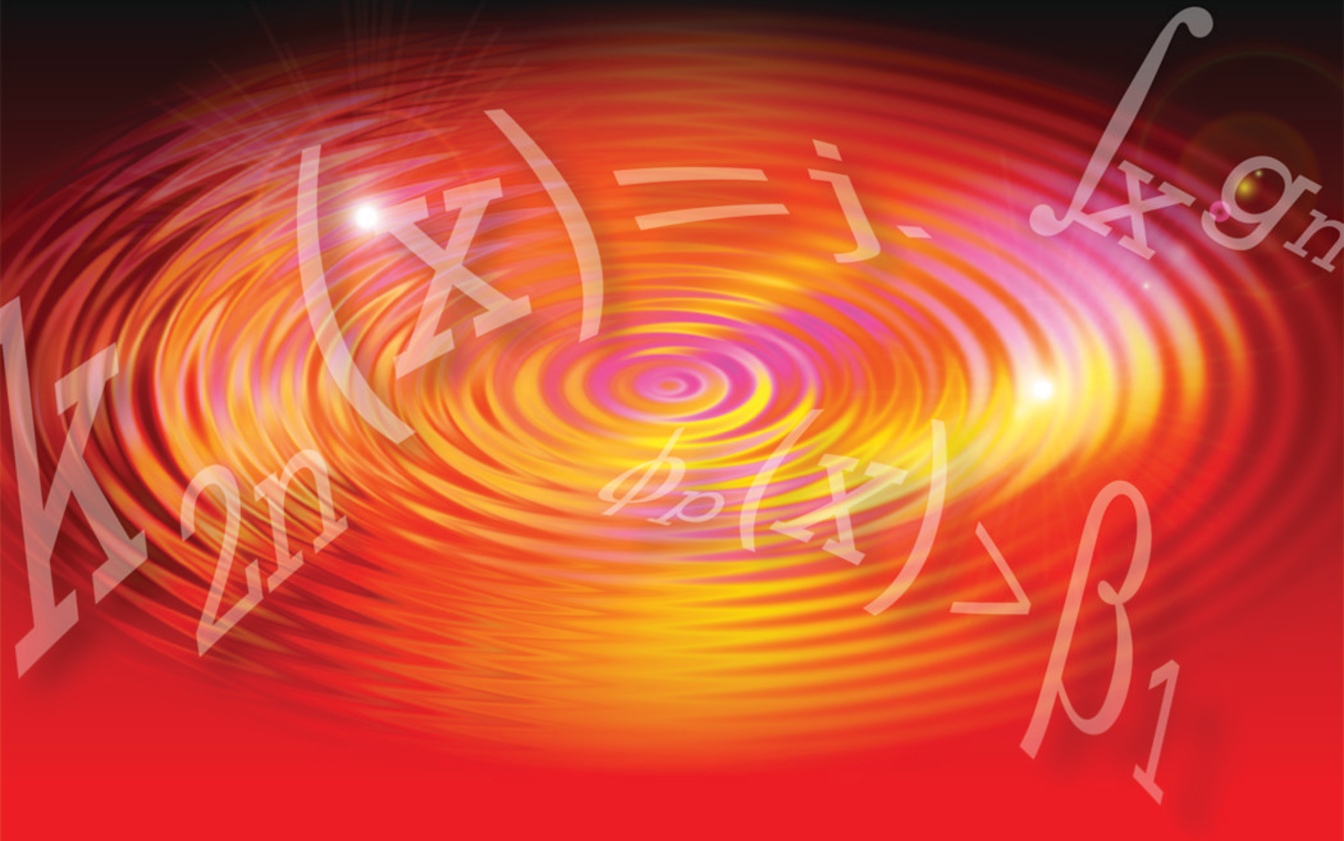
Quasi-invariant and Pseudo-differentiable Measures in Banach Spaces



Sergey V. Ludkovsky

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Preface

This book is devoted to new results of investigations of non-Archimedean functional analysis, which is becoming more important nowadays due to the development of non-Archimedean mathematical physics, particularly, quantum mechanics, quantum field theory, theory of super-strings and supergravity [VV89, VVZ94, ADV88, Cas02, DD00, Ish84, Khr90, Lud99t, Lud03b, Mil84, Jan98]. Recently non-Archimedean analysis was found to be useful in dynamical systems, mathematical biology, mathematical psychology, cryptology and information theory. On the other hand, quantum mechanics is based on measure theory and probability theory. The results of this book published mainly in papers [Lud02a, Lud03s2, Lud04a, Lud96c, Lud99a, Lud00a, Lud99t, Lud01f, Lud00f, Lud99s, Lud04b] have served for investigations of non-Archimedean stochastic processes [Lud0321, Lud0341, Lud0348, Lud01f, LK02]. Stochastic approach in quantum field theory is actively used and investigated especially in recent years (see, for example, and references therein [AHKMT93, AHKT84]). As it is well-known in the theory of functions great role is played by continuous functions and differentiable functions.

In the classical measure theory the analog of continuity is quasi-invariance relative to shifts and actions of linear or non-linear operators in the Banach space. Moreover, differentiability of measures is the stronger condition and there is very large theory about it in the classical case. Apart from it the non-Archimedean case was less studied. Since there are not differentiable functions from the p-adic field $\mathbf{Q_p}$ into $\mathbf{R}$ or into another p'-adic non-Archimedean field $\mathbf{Q}_{p'}$ with $p \neq p'$, then instead of differentiability of measures their pseudo-differentiability is considered.

Traditional or classical mathematical analysis and functional analysis work mainly over the real and complex fields. But there are well-known many other infinite fields with non-trivial multiplicative norms since the end of the 19-th century. If a multiplicative norm in a field $\mathbf{K}$ or a norm in a vector space X over $\mathbf{K}$ satisfies instead of the triangle inequality stronger condition: $|x+y| \leq \max(|x|,|y|)$ for each $x, y \in \mathbf{K}$ or in X respectively, then it is called the non-Archimedean norm. Such fields and vector spaces with non-Archimedean norms are frequently called for short non-Archimedean fields and non-Archimedean normed spaces correspondingly. Therefore, mathematical analysis and functional analysis over non-Archimedean fields develop already during rather long period of time, but they remain substantially less elaborated in comparison with that of the classical one.

The first chapter of this book is devoted to real-valued measures and in the second chapter measures with values in non-Archimedean fields are described. Though the results of these two chapters have served in investigations of quasi-invariant and pseudo-differentiable measures on topological totally disconnected groups which can be non-locally compact such as Lie groups, diffeomorphism groups and geometric wrap or loop groups of Banach manifolds over non-Archimedean fields. They also were used for investigations of representations of such groups in the series of papers [Lud99a, Lud00a, Lud99t, Lud98b, Lud01s, Lud02b, Lud0348, Lud08] (for comparison, in the case of non-locally compact groups over **R** or **C** see, for example, also [VGG75, PS68, AHKMT93, AHKT84, Kos94, Shim94, Lud99r, Lud01f]). But restricting by the scope of this book only few results illustrating applications of such measures are given in Chapters 3 and 4, which provide main differences between the case of locally compact groups and non-locally compact groups.

Quasi-invariance and pseudo-differentiability of measures can also be used for studying properties of transition conditional measures of stochastic processes, for solution of pseudo-differential or anti-derivational stochastic equations.

In the case of locally compact groups there is possible to construct a nontrivial Haar measure on a group and this serves to define C^*-algebra corresponding to this group. This is the crucial point in investigations of their representations in linear spaces and finding invariant closed linear subspaces [Nai68, FD88]. In particular, decomposition of unitary representations in complex Hilbert spaces into a direct integral of topologically irreducible representations can be accomplished with the help of this technique.

In the case of non-locally compact groups there does not exist any nontrivial Haar measure, but only a measure quasi-invariant relative to left (or right) shifts by elements of a proper subgroup. Then it is possible to associate with the latter measure an algebra over such group, but it is not the C^*-algebra, its structure is much more complicated, though for it there is proved the analog of the Gelfand-Naimark theorem in Chapter 3 for real-valued measures and its non-Archimedean counterpart for non-Archimedean valued measures in Chapter 4.

Effective ways to use quasi-invariant and pseudo-differentiable measures are given in the cited above articles of the author. Professor I.V. Volovich had been discussing with me the matter and interested in results of my investigations of non-Archimedean analogs of Gaussian measures such as to satisfy as many Gaussian properties as possible as he has planned to use such measures in non-Archimedean quantum field theory. The question was not so simple. He has supposed that properties with mean values, moments, projections, distributions and convolutions of such measures can be considered analogously. This matter we had been debating with Professor B. Diarra, who had doubted that all Gaussian circumstances can be fulfilled. But thorough analysis has shown, that not all properties can be satisfied, because in such case the linear space would have a structure of the **R**-linear space (see §I.6 and §II.6). Nevertheless, many of the properties there is possible to satisfy in the non-Archimedean case also. Gaussian measures are convenient to work in the classical case, but in the non-Archimedean case they do not play so great role.

Strictly speaking no any nontrivial Gaussian measure exists in the non-Archimedean case, but measures having few properties analogous to that of Gaussian can be outlined. Supplying them with definite properties depends on a subsequent task for which problems they may be useful. For example, if each projection μ_Y of a measure μ on a finite dimensional subspace Y over a field **K** is equivalent to the Haar measure λ_Y on Y, then this is very well property. But in the classical case, as it is well-known, such property does not imply that the measure μ is Gaussian, since each measure $\nu_Y(dx) = f(x)\lambda_Y(dx)$ with

$f \in L^1(Y, \lambda_Y, \mathbf{R})$ is absolutely continuous relative to the Lebesgue measure λ_Y on Y and this does not imply Gaussian properties of moments or its characteristic functional (see, for example, [GV61, DF91, HT74, VTC85]). The class of measures having such properties of projections is much wider than that of Gaussian and is described by the Kolmogorov and Kakutani theorems.

This book is devoted to general theory of quasi-invariant and pseudo-differentiable measures not restricting the theory by a particular class of measures. Earlier versions of the results about quasi-invariant and pseudo-differentiable measures were communicated to A.C.M. van Rooij and W.H. Schikhof (Nijmegen) in 1994-1995 and also to M. van der Put (Groningen), who were interested in it. This text was thoroughly read by B. Diarra (Clermont-Ferrand) who has recommended to write about real-valued measures and measures with values in non-Archimedean fields separately, because their theory differ substantially. Results of these investigations were also communicated at research seminars and lectures of Mathematical Department of Blaise Pascal University in Clermont-Ferrand and at Department of Mathematical Physics of Steklov Mathematical Institute in Moscow and at Chair of Higher Algebra at Mathematical Department of Moscow State University (M.V. Lomonosov), at the conference by non-Archimedean analysis at Växjö University (Sweden).

The starting point for this work was specific non-Archimedean general measure theory of A.P. Monna and T.A. Springer, A.C.M. van Rooij and W.H. Schikhof, etc. Also some results of investigations of V.S. Vladimirov, I.V. Volovich, E.I. Zelenov and A.Yu. Khrennikov were used, who considered it for problems of non-Archimedean quantum mechanics. More detailed discussions of sources and somewhat related works are given in comments and introductions to each chapter.

This book is devoted to more specific measure theory of quasi-invariant and pseudo-differentiable measures in Banach spaces including infinite-dimensional over fields. The author has written this theory in details, though it also opens ways for further investigations in this new area. The results of this book provide also wider classes of quasi-invariant and pseudo-differentiable measures on non-locally compact groups and non-Archimedean manifolds with the help of approaches described in papers on groups and manifolds cited above.

In the first chapter quasi-invariant and pseudo-differentiable measures on a Banach space X over a non-Archimedean locally compact infinite field with a non-trivial normalization are defined and constructed and studied. Measures are considered with values in $\mathbf{R}$. Theorems and criteria are formulated and proved about quasi-invariance and pseudo-differentiability of measures relative to linear and non-linear operators on X. Characteristic functionals of measures are studied. Moreover, the non-Archimedean analogs of the Bochner-Kolmogorov and Minlos-Sazonov theorems are proved. Convolutions of measures and infinite products of measures also are considered. Convergence of quasi-invariant and pseudo-differentiable measures in the corresponding spaces of measures is investigated.

In the second chapter measures are considered with values in non-Archimedean fields, for example, the field $\mathbf{Q_p}$ of p-adic numbers. Classes of quasi-invariant and pseudo-differentiable measures on a Banach space X over a non-Archimedean locally compact infinite field with a non-trivial normalization are described and studied. Their quasi-invariance and pseudo-differentiability relative to linear operators and non-linear transformations on

X is investigated. The corresponding theorems are demonstrated. The important instrument for analysis of the measure is its characteristic functional. Therefore, characteristic functionals of a measure are studied below. Other important cornerstones are the non-Archimedean analogs of the Bochner-Kolmogorov and Minlos-Sazonov theorems, which are investigated as well. Moreover, infinite products of measures are considered and the analog of the Kakutani theorem is proved. Then spaces of measures are studied and a convergence of nets of quasi-invariant and pseudo-differentiable measures is investigated.

In the third and the fourth chapters properties of quasi-invariant measures (real and non-Archimedean valued respectively) relative to dense subgroups on topological groups are considered. There predominantly non-locally compact groups are major objects such as

(i) a group of diffeomorphisms $Diff(t,M)$ of non-Archimedean manifold M in cases of locally compact and non-locally compact M, where t is a class of smoothness,

(ii) a general Banach-Lie group over a classical or non-Archimedean field,

(iii) wrap or loop groups of real and non-Archimedean manifolds.

The main feature there consists in that algebras of quasi-invariant measures are introduced and studied. Generally these algebras appear to be non-commutative and non-associative over non-locally compact groups. Therefore, they do not induce any C^*-algebra and the Gelfand-Mazur theorem is not valid already for such algebras. Nevertheless, for them analogs of the Gelfand-Mazur theorem are proved. One may mention that the non-local compactness causes a twisted algebraic structure of measure spaces. This situation can be compared with representation theory of groups in non-Archimedean linear spaces. Over infinite fields with non-trivial non-Archimedean multiplicative norms the aforementioned theorem is not accomplished due to existence of transcendental extensions of such fields (see [Roo78] and references therein).

For reading of this book it is better to have some basic knowledge of the material contained in works [Roo78, Sch84, DF91, Eng86, FD88], though below all necessary definitions and notations are given. The present book can be used for studying of this part of functional analysis, for reference and for further investigations. Its results can be used not only in Mathematics in functional analysis, theory of topological and Lie groups and their representations, topological algebras, dynamical systems, probability theory, random functions and stochastic processes and equations, integral transforms, but also in quantum mechanics and quantum field theory, mathematical biology, mathematical psychology, cryptology, information theory, etc.

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Notation

$Af(X,\mu)$ §I.2.1, II.2.1;
$Bco(X)$ §II.2.1;
$Bf(X)$ §I.2.1;
$B(X,x,r)$ §I.2.2;
B_{+}, C_{+} §I.2.29;
$c_0(\alpha,\mathbf{K})$, P_L §I.2.2;
$C(X,\mathbf{K})$ §I.2.6;
$\hat{\mathsf{C}}(Y,\Gamma)$, $\tau(Y)$ §I.2.26;
$\mathsf{C}(Y,\Gamma)$, $\tau(Y)$ §II.2.20;
$\mathbf{C_s}$ Introduction of Chapter II;
$c_0(\{H_i : i \in \mathbf{N_o}\})$ §4.17;
δ_0 §I.2.8;
$\mathbf{F_p}$ Introduction of Chapter I;
$\mathbf{F_p}(\theta)$ Introduction of Chapter I;
$\mathbf{K}$ Introduction of Chapter I;
$\mathbf{K_s}$ Introduction of Chapter II;
$L(X,\mu,\mathbf{K_s})$ §II.2.4;
$l_2(\{H_i : i \in \mathbf{N_o}\})$ §3.16;
$\mathsf{M}(X)$ §I.2.1, II.2.1;
$\mathsf{M}_t(X)$ §I.2.1, II.2.1;
μ_L §I.2.2, §II.2.2;
$\{\mu_{L_n} : n\}$ §I.2.2, §II.2.2;
$\mu_1 * \mu_2$ §I.2.11, §II.2.8;
$\nu \ll \mu$, $\nu \sim \mu$, $\nu \perp \mu$ §I.2.36, II.2.31;
$\mathbf{Q_p}$ Introduction of Chapter I;
$\mathbf{Q_s}$ Introduction of Chapter II;
$\mathbf{U_s}$ Introduction of Chapter II and §II.4.1;
$\psi_{q,\mu}$ §I.2.14;
$\theta(z) = \hat{\mu}$ §I.2.6, §II.2.5;
χ_ξ §I.2.6, II.2.5.

Chapter 1

Real-Valued Measures

1.1. Introduction

There are known works on integration in a Banach space over the field **R** of real numbers or the field **C** of complex numbers [Bou63-69, Chr74, Con84, DF91, Sko74, VTC85]. But for a non-Archimedean Banach space X, which is over a field with a non-Archimedean multiplicative norm, this theory is less developed. Integration theory in X is a very important part of the non-Archimedean analysis. The period is such that the advances of quantum mechanics and different branches of modern physics related, for example, with theories of elementary particles lead to the necessity of developing integration theory in a non-Archimedean Banach space [ADV88, DD00, Ish84, Mil84, VVZ94, Jan98]. Certainly, it may also be useful for the development of non-Archimedean functional analysis.

As it is well-known non-Archimedean analysis develops rapidly in recent years and has many principal differences from the classical analysis [Khr90, Roo78, Sch84, Sch89, Sch71, VVZ94]. Linear topological spaces over non-Archimedean fields are totally disconnected. Therefore, classes of smoothness for functions and compact operators are defined for them quite differently from that of the classical case. In such spaces also the notion of the orthogonality of vectors has obtained quite another meaning. We mention also that in the non-Archimedean case the Radon-Nikodym theorem and the Lebesgue theorem about convergence are not true in the classical form, but their analogs are true under more rigorous and another conditions. Very strong differences are for measures with values in non-Archimedean fields in comparison with that of with real- or complex-valued, because classical notions of σ-additivity and quasi-invariance have lost their meaning.

Nonetheless, the development of the non-Archimedean functional analysis and its applications in non-Archimedean quantum mechanics [VVZ94, Khr90, Jan98] leads to the necessity of solving such problems. Frequently advances of quantum mechanics on manifolds and quantum field theory are related with diffeomorphism groups and wrap or loop groups, their representations and measures on them [Ish84, Mil84, Lud99t, Lud00a]. In publications [Lud96, Lud98s, Lud99t, Lud98b, Lud00a] quasi-invariant measures on diffeomorphism and wrap or loop groups and also on manifolds were constructed. Then such measures were used for the investigation of unitary including irreducible representations in complex Hilbert spaces of topological and Lie groups [Lud99t, Lud98b, Lud99a]. The theorems demonstrated in this book enlarge classes of measures on such groups and manifolds. As the consequence this also enlarges classes of available representations. For example, theorems analogous to that of the Minlos-Sazonov type characterize measures with the help of characteristic functionals and compact operators. Compact operators are more useful in the non-Archimedean case, rather than nuclear operators in the classical case. Of exceptional importance are theorems of the Bochner-Kolmogorov and Kakutani type characterizing products of measures and their absolute continuity relative to others measures.

In this chapter measures are considered on Banach spaces, though the results given below can be developed for more general topological vector spaces, for example, it is possible to follow the ideas of works [Mad91c, Mad91a, Mad85], in which were considered non-Archimedean analogs of the Minlos-Sazonov theorems for real-valued measures on topological vector spaces over non-Archimedean fields of zero characteristic. But it is impossible to make in one chapter or book. In this book, apart from articles of Mądrecki, measures are considered also with values in non-Archimedean fields (see Chapter 2). For the cases of real-valued measures also Banach spaces over non-Archimedean fields $\mathbf{K}$ of characteristic $char(\mathbf{K}) > 0$ are considered. Recall that a real-valued measure m on a locally compact Hausdorff totally disconnected Abelian topological group G is called the Haar measure, if

(H) $m(x+A) = m(A)$ for each $x \in G$ and each Borel subset A in G. It is useful to start from measures on the field $\mathbf{K}$ equivalent to the Haar measure up to a multiplier which is a measurable function on $\mathbf{K}$.

Many of definitions and theorems described below are the non-Archimedean analogs of classical results. But frequently their formulations and proofs differ strongly. The reader, need not be familiar with the part of classical functional analysis about measures in infinite-dimensional spaces, because this book contains all necessary definitions and results.

Below in §2 sequences of weak distributions, characteristic functions of measures and their properties are defined and investigated. The non-Archimedean analogs of the Minlos-Sazonov and Bochner-Kolmogorov theorems are presented. For this quasi-measures also are considered, because they are tightly related with measures. In §3 products of measures are described together with their density functions. In the present chapter broad classes of quasi-invariant measures are defined and constructed. Further theorems about quasi-invariance of measures under definite linear and non-linear transformations $U : X \to X$ are demonstrated. §4 contains a notion of pseudo-differentiability of measures. This is very important, because for functions $f : \mathbf{K} \to \mathbf{R}$ there is not any notion of differentiability, that is there is not such non-linear non-trivial function f. Then criteria for the pseudo-differentiability are studied. In §5 theorems about converegence of measures there are given with taking into account their quasi-invariance and pseudo-differentiability, that is, in the corresponding spaces of measures. The main results are Theorems 2.27, 2.35, 3.4, 3.20, 3.24, 3.25, 4.2, 4.3, 4.5, 4.7, 5.7-5.10.

The first chapter tackles real-valued measures. Below in the second chapter measures with values in non-Archimedean fields are considered as well. This is caused by differences in definitions, formulations of statements and their proofs in two such principally distinct cases. To avoid repeating when differences are small only briefly matters are discussed in the second chapter and refereing the first.

Notations. Henceforth, $\mathbf{K}$ denotes a locally compact infinite field with a non-trivial norm, then the Banach space X is over $\mathbf{K}$. In the present chapter measures on X have values in the real field **R**. We assume that either **K** is a finite algebraic extension of the field of p-adic numbers $\mathbf{Q_p}$ or $char(\mathbf{K}) = p$. In the latter case **K** is isomorphic with the field $\mathbf{F_p}(\theta)$ of formal power series consisting of elements $x = \sum_{j=N}^{\infty} a_j \theta^j$, where $a_j \in \mathbf{F_p}$, $|\theta| = p^{-1}$, $\mathbf{F_p}$ is the finite field of p elements, p is a prime number, $N = N(x) \in \mathbf{N}$, **N** denotes the set of natural numbers $\mathbf{N} = \{1,2,3,\dots\}$. If $x \neq 0$, $a_N \neq 0$, then $|x| = p^{-N}$, certainly $|0| = 0$. Mention that $\mathbf{Q_p}$ is of zero characteristic $char(\mathbf{Q_p}) = 0$. Recall that each non-zero p-adic number x can be written in the form $x = \sum_{j=N}^{\infty} x_j p^j$, where $x_j \in \{0,1,\dots,p-1\}$ for each j, while $x_N \neq 0$ and $N = N(x) \in \mathbf{N}$. The norm of such x is $|x| := p^{-N}$, while $|0| = 0$. An equivalent multiplicative norm is $|x|^c$ with a marked positive constant $c > 0$.

These imply that the locally compact field **K** has the Haar measures with values in **R** [Roo78]. If X is a Hausdorff topological space with a small inductive dimension $ind(X) = 0$ (see [Eng86]), then E denotes an algebra of subsets of X. As a rule we use $E \supset Bf(X)$ for real-valued measures, where $Bf(X)$ is a Borel σ-field of X in §2.1, $Af(X,\mu)$ is the completion of E by a measure μ in §2.1.

Remind that a topological space X is called zero-dimensional, if X is a non-void T_1-space having a base of topology consisting of clopen subsets, where a subset U in X is called clopen if it is closed and open simultaneously. Then each zero-dimensional space is regular (Tychonoff) space.

For any subset A of a topological space X a boundary of A is defined as $Fr\,A := cl(A) \cap cl(X \setminus A) = [cl(A)] \setminus Int(A)$, where $cl(A)$ denotes the closure of A in X and $Int(A)$ is the interior of A in X. Suppose that X is a regular space and n is a non-negative integer, $0 \leq n \in \mathbf{Z}$. Put

$(MU1)$ $ind(X) = -1$ if and only if $X = \emptyset$;

$(MU2)$ $ind(X) \leq n$ if for each point $x \in X$ and every its neighborhood V an open subset $U \subset X$ exists such that $x \in U \subset V$ and $ind(Fr\,U) \leq n-1$;

$(MU3)$ $ind(X) = n$ if $ind(X) \leq n$ and the inequality $ind(X) \leq n-1$ is not satisfied;

$(MU4)$ $ind(X) = \infty$ if the inequality $ind(X) \leq n$ is not accomplished for any n.

The number $ind(X)$ is called the small inductive dimension of the topological space X.

1.2. Distributions and Families of Measures

2.1. For a Hausdorff topological space X with a small inductive dimension $ind(X) = 0$ [Eng86] the Borel σ-field we denote by $Bf(X)$, where Bf stands as the abbreviation from two words. Henceforth, measures μ are given on a measurable space (X,E), where E is a σ-algebra of subsets in X. The completion of E relative to μ we denote by $Af(X,\mu)$. The total variation of a measure μ with values in **R** on a subset A is denoted by $\|\mu|_A\|$ or $|\mu|(A)$ for $A \in Af(X,\mu)$. Recall that

$$|\mu|(A) := \sup_{\pi} \sum_{E \in \pi} |\mu(E)|$$

it is first defined on E and then is extended on the completion for each $A \in Af(X,\mu)$, where π is an arbitrary finite partition of a set A.

If μ is non-negative and $\mu(X) = 1$, then it is called a probability measure.

A measure μ on E is called Radon, if for each $\varepsilon > 0$ there exists a compact subset $C \subset X$ such that $\|\mu|_{(X\setminus C)}\| < \varepsilon$. Henceforth, $\mathsf{M}(X)$ denotes a space of norm-bounded measures $\|\mu\| = |\mu|(X) < \infty$, $\mathsf{M}_t(X)$ is its subspace of Radon norm-bounded measures.

Remind also the following facts.

I. Definitions. A family of subsets $\{S_n : n \in \Upsilon\}$ in a set X is called shrinking, if for each $k,n \in \Upsilon$ there exists $m \in \Upsilon$ so that $S_m \subset S_k \cap S_n$, where Υ is some set.

An ultra-metric space (X,d) is called spherically complete, if every shrinking sequence $\{B_n : n \in \mathbf{N}\}$ of balls in X has a non-void intersection. By a spherical completion $< F,T >$ of a normed $\mathbf{K}$-vector space E the pair $< F,T >$ is so called consisting of a spherically complete space F and a $\mathbf{K}$-linear isometry $T : E \to F$ such that F has not a proper spherically complete linear subspace containing in itself $T(E)$. The spherical completion is frequently denoted simply also by F instead of $< F,T >$.

For a normed $\mathbf{K}$-linear space E two vectors $x,y \in E$ are called orthogonal, if $\|ax+by\| = \max(\|ax\|,\|by\|)$ for all $a,b \in \mathbf{K}$. For a real number $0 < t \le 1$ a finite or an infinite sequence of elements $x_j \in E$ is called t-orthogonal, if $\|a_1x_1 + \dots + a_mx_m + \dots\| \ge t\max(\|a_1x_1\|,\dots,\|a_mx_m\|,\dots)$ for each $a_1,\dots,a_m,\dots \in \mathbf{K}$ with $a_1x_1 + \dots + a_mx_m + \dots \in E$.

We say, that a Banach space E over a field $\mathbf{K}$ complete relative to its non-Archimedean multiplicative norm has the countable type, if E is a closed $\mathbf{K}$-linear span of some countable its subset.

If Z is a $\mathbf{K}$-linear subspace of the normed space E over $\mathbf{K}$, then E is called the immediate extension of Z, if 0 is an unique element in E orthogonal to the subspace Z.

Let ω be a non-void set and there is given a function $s : \omega \to (0,\infty)$. For a function $f : \omega \to \mathbf{K}$ put

$$\|f\|_s := \sup\{|f(x)|s(x) : x \in \omega\}.$$

The set of all functions $f : \omega \to \mathbf{K}$, for which the norm $\|f\|_s$ is finite, forms the $\mathbf{K}$-vector space denoted by $l^\infty(\mathbf{K},\omega;s)$, which is the Banach space relative to the norm $\|*\|_s$.

By $c_0(\mathbf{K},\omega;s)$ we denote the closed subspace in $l^\infty(\mathbf{K},\omega;s)$, consisting from all functions f, satisfying the additional condition: for each $b > 0$ the set $\{x \in \omega : |f(x)|s(x) > b\}$ is finite. If a function s takes only one fixed value, for example, the unit, then we omit s from the notation of the Banach space.

Further we shall need two theorems.

II. Theorem (5.13 [Roo78]). *For each Banach space E over the field $\mathbf{K}$ complete relative to its non-Archimedean multiplicative norm the following conditions are equivalent:*

(α). *Each maximal orthogonal system of elements in E is a basis.*

(β). *Each closed $\mathbf{K}$-linear subspace in E has an orthogonal complement.*

(γ). *Each closed $\mathbf{K}$-linear subspace in E of countable type has an orthogonal complement.*

(δ). *E is not an immediate extension of of any its proper closed $\mathbf{K}$-linear subspace in E.*

(ε). *Each countable orthogonal system of elements in E can be extended up to an orthogonal basis in E.*

III. Theorem (5.16 [Roo78]. *For an infinite-dimensional Banach space E over the field $\mathbf{K}$ complete relative to its non-Archimedean multiplicative norm conditions $(\alpha - \varepsilon)$ from Theorem II are equivalent with any of the following conditions $(\zeta - \iota)$:*

(ζ). *Each closed $\mathbf{K}$-linear subspace in E is spherically complete.*

(η). *E has an orthogonal basis and is spherically complete.*

(θ). *Each strictly decreasing sequence of values of a norm in E converges to zero.*

(ι). *The normalization group* $\Gamma_{\mathbf{K}} := \{|x| : x \in \mathbf{K}, x \neq 0\}$ *of the field* $\mathbf{K}$ *is discrete in* $(0,\infty)$. *For a number* $\pi \in \mathbf{K}$ *so that* $0 < |\pi| < 1$ *and* $|\pi|$ *is the generator of the normalization group* $\Gamma_{\mathbf{K}}$ *of the field* $\mathbf{K}$ *there exists a set* ω *and a function* $s : \omega \to (|\pi|, 1]$ *so that E is isometrically* $\mathbf{K}$*-linearly isomorphic with* $c_0(\mathbf{K}, \omega; s)$, *where the set of values of a function s is well ordered.*

If E is infinite-dimensional and of countable type over $\mathbf{K}$, *then conditions* $(\alpha - \iota)$ *are equivalent to the spherical completeness of the Banach space E.*

2.2. As usually denote by $c_0(\alpha,\mathbf{K})$ the Banach space $c_0(\alpha,\mathbf{K}) := \{x : \ x = (x_j : j \in \alpha), card(j : |x_j|_{\mathbf{K}} > b) < \aleph_0$ for each $b > 0\}$, where α is a set considered as an ordinal due to the Kuratowski-Zorn lemma, $card(A)$ denotes the cardinality of A, the norm is $\|x\| := \sup(|x_j| : j \in \alpha)$. A dimension of X over $\mathbf{K}$ is by the definition $dim_{\mathbf{K}} X := card(\alpha)$. Each Banach space X over $\mathbf{K}$ in view of Theorems 5.13 and 5.16 [Roo78] is $\mathbf{K}$-linearly topologically isomorphic with $c_0(\alpha,\mathbf{K})$, since the field $\mathbf{K}$ is spherically complete. For each closed $\mathbf{K}$-linear subspace L in X there exists an operator of a projection $P_L : X \to L$. Moreover, an orthonormal in the non-Archimedean sense basis in L has a completion to an orthonormal basis in X such that P_L can be defined in accordance with a chosen basis.

If $A \in Bf(L)$, then $P_L^{-1}(A)$ is called a cylindrical subset in X with a base A, $B^L := P_L^{-1}(Bf(L))$, $B_0 := \cup(B^L : L \subset X, L$ is a Banach subspace $, dim_{\mathbf{K}} X < \aleph_0)$. The minimal σ-algebra σB_0 generated by B_0 coincides with $Bf(X)$, if $dim_{\mathbf{K}} X \leq \aleph_0$. Henceforward, it is assumed that $\alpha \leq \omega_0$, where ω_0 is the initial ordinal with the cardinality $\aleph_0 := card(\mathbf{N})$. This implies that there exists an increasing sequence of Banach subspaces $L_n \subset L_{n+1} \subset \cdots$ such that $cl(\cup[L_n : n]) = X$, $dim_{\mathbf{K}} L_n = \kappa_n$ for each n, where $cl(A) = \bar{A}$ denotes the closure of A in X for $A \subset X$.

We fix a family of projections $P_{L_n}^{L_m} : L_m \to L_n$ so that $P_{L_n}^{L_m} P_{L_k}^{L_n} = P_{L_k}^{L_m}$ for all $m \geq n \geq k$. Projections of the measure μ onto linear subspaces L denoted by $\mu_L(A) := \mu(P_L^{-1}(A))$ for each $A \in Bf(L)$ compose the consistent family:

$$\mu_{L_n}(A) = \mu_{L_m}(P_{L_n}^{-1}(A) \cap L_m) \tag{1}$$

for each $m \geq n$, since there are projectors $P_{L_n}^{L_m}$, where $\kappa_n \leq \aleph_0$ and there may be chosen $\kappa_n < \aleph_0$ for each n.

An arbitrary family of measures $\{\mu_{L_n} : n \in \mathbf{N}\}$ having property (1) is called a sequence of weak distributions (see also [DF91, Sko74] in the classical case).

By $B(X,x,r)$ we denote the ball $\{y : \ y \in X, \ \|x - y\| \leq r\}$, which is clopen (closed and open) in X.

2.3. Lemma. *A sequence of weak distributions* $\{\mu_{L_n} : n\}$ *is generated by some measure* μ *on* $Bf(X)$ *if and only if for each* $c > 0$ *there exists* $b > 0$ *such that*

$$||\mu_{L_n}|(B(X,0,r) \cap L_n) - |\mu_{L_n}|(L_n)| \leq c$$

for each $r \geq b$ *and every* $n \in \mathbf{N}$ *and*

$$\sup_n |\mu_{L_n}|(L_n) < \infty.$$

Proof. We consider the non-trivial case, when μ_{L_n} are non-zero. In the case of μ with values in $\mathbf{R}$ we can use the Jordan decomposition $\mu = \mu^+ - \mu^-$ of the measure μ, where μ^+ and μ^- are non-negative measures so that $\mu^+(A) := \sup_{B\subset A, B\in E} \mu(B)$ and $\mu^-(A) := -\inf_{B\subset A, B\in E} \mu(B)$ and $|\mu|(A) = \mu^+(A) + \mu^-(A)$ for $A \in E$, since X is the Radon space in view of Theorem 1.2 §I.1.3 [DF91].

Recall that the class $\mathcal{K}_X$ of all countably compact subsets in X is compact, that is for each sequence K_n in $\mathcal{K}_X$ with $\bigcap_{n=1}^{\infty} K_n = \emptyset$ there exists a natural number $m \in \mathbf{N}$ so that $\bigcap_{n=1}^{m} K_n = \emptyset$. A topological space X is called countably compact if from each countable open covering of X it is possible to choose a finite sub-covering of X [Eng86]. A measure μ on X is called Radon if it is approximated from below by the class $\mathcal{K}_X$, that is for each $A \in E$ and every $\varepsilon > 0$ there exists $K \in \mathcal{K}_X$ such that $|\mu|(A\setminus K) < \varepsilon$.

Therefore, reduce the proof to the variant, when the measure μ is non-negative and $\mu(X) = 1$, since for $0 < \mu(X) < \infty$ substitute $\mu(A)$ on $\mu(A)/\mu(X)$ in case of necessity.

If a sequence of weak distributions $\{\mu_{L_n}\}$ is generated by the measure μ, then for $0 < \varepsilon < 1$ choose $r > 0$ such that $\mu(B(X,0,r)) > 1-\varepsilon$, since $\lim_{r\to\infty} \mu(B(X,0,r)) = 1$. Therefore, $\mu_L(B(X,0,R)\cap L) = \mu(P_L^{-1}(B(X,0,R)\cap L)) \ge \mu(B(X,0,R)) \ge \mu(B(X,0,r)) > 1-\varepsilon$ for each $R > r$.

Vice versa let a sequence of weak distributions satisfying conditions of the lemma be given. On the algebra B_0 define a finitely additive mapping $\mu(A) := \mu_L(A)$ for $A \in B^L$.

Consider a cylindrical set A_n with a base B_n in L_n so that $\bigcap_n A_n = \emptyset$ and $A_{n+1} \subset A_n$ for each n. For each n choose a closed set C_n such that $C_n \subset B_n$ and $\mu_{L_n}(B_n \setminus C_n) < \varepsilon_n$. Then take $D_n := \bigcap_{m=1}^{n} \{x : P_{L_m}x \in C_m\} \cap L_n$, hence D_n is closed and

$$\mu_{L_n}(B_n \setminus D_n) \le \sum_{m=1}^{n} \mu_{L_m}(B_m \setminus C_m) \le \sum_{m=1}^{n} \varepsilon_m.$$

For sets $J_n := P_{L_n}^{-1}(D_n)$ we get $J_{n+1} \subset J_n$ and $\bigcap_n J_n = \bigcap_n A_n$ and $\mu(A_n) \le \mu(J_n) + \sum_{m=1}^{n} \varepsilon_m$. If $\lim_{n\to\infty} \mu(J_n) = 0$, then $\lim_{n\to\infty} \mu(A_n) = 0$, since $\sum_{m=1}^{n} \varepsilon_m$ can be taken arbitrary small.

Therefore, we can consider the variant when each base B_n is closed. Hence each A_n is weakly closed in X, since the field $\mathbf{K}$ is sphericallty complete and the toplogically dual spoace of all continuous $\mathbf{K}$-linear functionals on X separates points in X. Then for each $0 < r < \infty$ the closed ball $B(X,x,r)$ is weakly closed and weakly compact, since $\mathbf{K}$ is locally compact. Since $B(X,0,R) \cap \bigcap_{n=1}^{\infty} A_n = \emptyset$, then $\bigcap_{n=1}^{\infty} (B(X,0,R) \cap A_n) = \emptyset$. Then there exists a natural niumber n so that $B(X,0,R) \cap A_n = \emptyset$, since each such set $B(X,0,R) \cap A_n$ is weakly compact and $A_{n+1} \subset A_n$ for each n. Thus $\mu(A_n) = \mu_{L_n}(A_n) \le \mu_{L_n}(L_n) - \mu_{L_n}(L_n \cap B(X,0,R)) \le \varepsilon$ for $r < R$. Since ε is arbitrary then $\lim_{n\to\infty} \mu(A_n) = 0$, consequently, μ is σ-additive.

2.4. Definition and notations. A function $\phi : X \to \mathbf{R}$ of the form $\phi(x) = \phi_S(P_S x)$ is called a cylindrical function if ϕ_S is a $Bf(S)$-measurable function on a finite-dimensional over $\mathbf{K}$ space S in X. For $\phi_S \in L^1(S,\mu,\mathbf{R})$ for μ with values in $\mathbf{R}$ we can define an integral by a sequence of weak distributions $\{\mu_{S_n} : n\}$:

$$\int_X \phi(x)\mu_*(dx) := \int_{S_n} \phi_{S_n}(x)\mu_{S_n}(dx),$$

where $L^p(S,\mu,\mathbf{R})$ denotes the space of all μ-measurable real-valued functions on S with finite norm $\|f\|_p := [\int_S |f(x)|^p |\mu|(dx)]^{1/p}$, where $1 \le p < \infty$.

2.5. Lemma. *A subset $A \subset X = c_0(\omega_0, \mathbf{K})$ is relatively compact if and only if A is bounded and for each $c > 0$ there exists a finite-dimensional over $\mathbf{K}$ subspace $L \subset X$ so that $\bar{A} \subset L^c := \{y \in X : d(y,L) := \inf\{\|x-y\| : x \in L\} \le c\}$.*

Proof. If A is bounded and for each $c > 0$ there exists L^c with $\bar{A} \subset L^c$, then there is a sequence $\{k(j) : j \in \mathbf{N}\} \subset \mathbf{Z}$ such that $\lim_{j\to\infty} k(j) = \infty$, $\bar{A} \subset \{x \in X : |x_j| \le p^{-k(j)}, j = 1,2,\dots\} =: S$, but X is Lindelöf, S is sequentially compact, consequently, $\bar{A}$ is compact (see §3.10.31 [Eng86]). If $\bar{A}$ is compact, then for each $c > 0$ there exists a finite number m such that $\bar{A} \subset \bigcup_{j=1}^m B(X, x_j, c)$, where $x_j \in X$. Therefore, $\bar{A} \subset L^c$ for $L = span_{\mathbf{K}}(x_j : j = 1,\dots,m) := (x = \sum_{j=1}^m b_j x_j : b_j \in K)$.

2.6. Remarks and definitions. When the characteristic of the field $\mathbf{K}$ is zero, $char(\mathbf{K}) = 0$, then $\mathbf{K}$ as the $\mathbf{Q_p}$ linear space is isomorphic with $\mathbf{Q_p^n}$, where $n \in \mathbf{N} := \{1,2,\dots\}$. The topologically dual space over $\mathbf{Q_p}$ (that is, of continuous linear functionals $f : \mathbf{K} \to \mathbf{Q_p}$) is isomorphic with $\mathbf{Q_p^n}$ [HR79]. For x and $z \in \mathbf{Q_p^n}$ we denote by (z,x) the following sum $\sum_{j=1}^n x_j z_j$, where $x = (x_j : j = 1,\dots,n)$, $x_j \in \mathbf{Q_p}$. Each number $y \in \mathbf{Q_p}$ has a decomposition $y = \sum_l a_l p^l$, where $\min(l : a_l \ne 0) =: ord_p(y) > -\infty$ ($ord(0) := \infty$) [NB85], $a_l \in (0,1,\dots,p-1)$, we define a symbol $\{y\}_p := \sum_{l<0} a_l p^l$ for $|y|_p > 1$ and $\{y\}_p = 0$ for $|y|_p \le 1$.

For a locally compact field $\mathbf{K}$ with a characteristic $char(\mathbf{K}) = p > 0$ let $\pi_j(x) := a_j$ for each $x = \sum_j a_j \theta^j \in \mathbf{K}$ (see Notation). For $\xi \in \mathbf{K}^*$ we denote $\xi(x)$ also by (ξ, x). All continuous characters $\chi : \mathbf{K} \to \mathbf{C}$ have the form

$$\chi_\xi(x) = \varepsilon^{z^{-1}\eta((\xi,x))} \tag{1}$$

for each $\eta((\xi,x)) \ne 0$, $\chi_\xi(x) := 1$ for $\eta((\xi,x)) = 0$, where $\varepsilon = 1^z$ is a root of unity, $z = p^{ord(\eta((\xi,x)))}$, $\pi_j : \mathbf{K} \to \mathbf{R}$, $\eta(x) := \{x\}_p$ and $\xi \in \mathbf{Q_p^{n*}} = \mathbf{Q_p^n}$ for $char(\mathbf{K}) = 0$, $\eta(x) := \pi_{-1}(x)/p$ and $\xi \in \mathbf{K}^* = \mathbf{K}$ for $char(\mathbf{K}) = p > 0$, $x \in \mathbf{K}$, (see §25 [HR79]). Each character χ is locally constant, hence $\chi : \mathbf{K} \to \mathbf{T}$ is also continuous, where $\mathbf{T}$ denotes the discrete group of all roots of 1 (by multiplication).

For a measure μ there exists a characteristic functional (that is, called the Fourier-Stieltjes transformation) $\theta = \theta_\mu : C(X,\mathbf{K}) \to \mathbf{C}$:

$$\theta(f) := \int_X \chi_e(f(x))\mu(dx), \tag{2}$$

where either $e = (1,\dots,1) \in \mathbf{Q_p^n}$ for $char(\mathbf{K}) = 0$, or $e = 1 \in \mathbf{K}^*$ for $char(\mathbf{K}) = p > 0$, $x \in X$, f is in the space $C(X,\mathbf{K})$ of continuous functions from X into $\mathbf{K}$, in particular for $z = f$ in the topologically dual (conjugate) space X^* over $\mathbf{K}$ of all continuous $\mathbf{K}$-linear functionals on X, $z : X \to \mathbf{K}$, $z \in X^*$, $\theta(z) =: \hat{\mu}(z)$. It has the following properties:

$$\theta(0) = 1 \text{ for } \mu(X) = 1 \tag{3a}$$

and $\theta(f)$ is bounded on $C(X,\mathbf{K})$;

$$\sup_f |\theta(f)| = 1 \text{ for probability measures}; \tag{3b}$$

$$\theta(z) \text{ is weakly continuous, that is, } (X^*, \sigma(X^*,X))\text{-continuous}, \tag{4}$$

$\sigma(X^*,X)$ denotes a weak topology on X^*, induced by the Banach space X over $\mathbf{K}$. To each $x \in X$ there corresponds a continuous linear functional $x^* : X^* \to \mathbf{K}$, $x^*(z) := z(x)$, moreover, $\theta(f)$ is uniformly continuous relative to the norm on

$$C_b(X,\mathbf{K}) := \{f \in C(X,\mathbf{K}) : \|f\| := \sup_{x\in X} |f(x)|_{\mathbf{K}} < \infty\};$$

$$\theta(z) \text{ is positive definite on } X^* \text{ and on } C(X,\mathbf{K}) \tag{5}$$

for μ with values in $[0,\infty)$.

Property (4) follows from Lemma 2.3, boundedness and continuity of χ_e and the fact that due to the Hahn-Banach theorem there is $x_z \in X$ with $z(x_z) = 1$ for $z \neq 0$ such that $z|_{(X\ominus L)} = 0$ and

$$\theta(z) = \int_X \chi_e(P_L(x))\mu(dx) = \int_L \chi_e(y)\mu_L(dy),$$

where $L = Kx_z$, also due to the Lebesgue theorem 2.4.9 [Fed69] for real measures. Indeed, for each $c > 0$ there exists a compact subset $S \subset X$ such that $|\mu|(X \setminus S) < c$, each bounded subset $A \subset X^*$ is uniformly equicontinuous on S (see (9.5.4) and Exer. 9.202 [NB85]), that is, $\{\chi_e(z(x)) : z \in A\}$ is the uniformly equicontinuous family (by $x \in S$). On the other hand, $\chi_e(f(x))$ is uniformly equicontinuous on a bounded $A \subset C_b(X,\mathbf{K})$ by $x \in S$.

Property (5) is accomplished, since

$$\sum_{l,j=1}^{N} \theta(f_l - f_j)\alpha_l\bar{\alpha}_j = \int_X |\sum_{j=1}^{N} \alpha_j\chi_e(f_j(x))|^2\mu(dx) \geq 0,$$

particularly, for $f_j = z_j \in X$, where $\bar{\alpha}_j$ is a complex conjugated number to α_j.

We call a functional θ finite-dimensionally concentrated, if there exists $L \subset X$, $dim_{\mathbf{K}}L < \aleph_0$, such that $\theta|_{(X\setminus L)} = \mu(X)$. For each $c > 0$ and $\delta > 0$ in view of Theorem I.1.2 [DF91] and Lemma 2.5 there exists a finite-dimensional over $\mathbf{K}$ subspace L and compact $S \subset L^\delta$ such that $\|X \setminus S\|_\mu < c$. Let $\theta^L(z) := \theta(P_L z)$.

This definition is correct, since $L \subset X$, X has the isometrical embedding into X^* as the normed space associated with the fixed basis of X, such that functionals $z \in X$ separate points in X. If $z \in L$, then $|\theta(z) - \theta^L(z)| \leq c \times b \times q$, where $b = \|X\|_\mu$, q is independent of c and b. Each characteristic functional $\theta^L(z)$ is uniformly continuous by $z \in L$ relative to the norm $\| * \|$ on L, since $|\theta^L(z) - \theta^L(y)| \leq |\int_{S'\cap L}[\chi_e(z(x)) - \chi_e(y(x))]$ $\mu_L(dx)| + |\int_{L\setminus S'}[\chi_e(z(x)) - \chi_e(y(x))]\ \mu_L(dx)|$, where the second term does not exceed $2C'$ for $\|L\setminus S'\|_{\mu_L} < c'$ for a suitable compact subset $S' \subset X$ and $\chi_e(z(x))$ is an uniformly equicontinuous by $x \in S'$ family relative to $z \in B(L,0,1)$.

Therefore,

$$\theta(z) = \lim_{n\to\infty} \theta_n(z) \tag{6}$$

for each finite-dimensional over $\mathbf{K}$ subspace L, where $\theta_n(z)$ is uniformly equicontinuous and finite-dimensionally concentrated on $L_n \subset X$, $z \in X$, $cl(\bigcup_n L_n) = X$, $L_n \subset L_{n+1}$ for every n, for each $c > 0$ there are n and $q > 0$ such that $|\theta(z) - \theta_j(z)| \leq cbq$ for $z \in L_j$ and $j > n$, $q = const > 0$ is independent from j, c and b. Let $\{e_j : j \in \mathbf{N}\}$ be the standard orthonormal in the non-Archimedean sense basis in X, $e_j = (0,\ldots,0,1,0,\ldots)$ with 1 in j-th place. Using countable additivity of μ, local constantness of χ_e, considering all $z = be_j$ and $b \in \mathbf{K}$, we get that $\theta(z)$ on X is non-trivial, whilst μ is a non-zero measure, since due to Lemma 2.3 μ is characterized uniquely by $\{\mu_{L_n}\}$. Indeed, for μ with values in $\mathbf{R}$ a measure μ_V on V, $dim_{\mathbf{K}}V < \aleph_0$, this follows from the properties of the Fourier transformation F on spaces of generalized functions and also on $L^2(V,\mu_V,\mathbf{C})$ (see §7 [VVZ94]), where

$$F(g)(z) := \lim_{r\to\infty}\int_{B(V,0,r)}\chi_e(z(x))g(x)m(dx),$$

$z\in V$, m is the Haar measure on V with values in $\mathbf{R}$. Therefore, the mapping $\mu\mapsto\theta_\mu$ is injective.

2.7. Proposition. *Let* $X=\mathbf{K}^{\mathbf{j}}$, $j\in\mathbf{N}$,

(a). *μ and ν be real probability measures on X, suppose ν is symmetric. Then* $\int_X\hat{\mu}(x)\nu(dx)=\int_X\hat{\nu}(x)\mu(dx)\in\mathbf{R}$ *and for each* $0<l<1$ *is accomplished the following inequality:*

$$\mu([x\in X:\quad \hat{\nu}(x)\le l])\le\int_X(1-\hat{\mu}(x))\nu(dx)/(1-l).$$

(b). *For each real probability measure μ on X there exists $r>p^3$ such that for each $R>r$ and $t>0$ the following inequality is accomplished:*

$$\mu([x\in X:\quad \|x\|\ge tR])\le c\int_X[1-\hat{\mu}(y\xi)]\nu(dy),$$

where $\nu(dx)=C\times\exp(-|x|^2)m(dx)$, *$m$ is the Haar measure on X with values in* $[0,\infty)$ *so that* $m(B(X,0,1))=1$, $\nu(X)=1$, $2>c=const\ge 1$ *is independent from t,* $c=c(r)$ *is non-increasing whilst r is increasing,* $C>0$.

Proof. (a). Recall that ν is symmetric, if $\nu(B)=\nu(-B)$ for each $B\in Bf(X)$. Therefore, $\int_X\chi_e(z(x))\nu(dx)=\int_X\chi_e(-z(x))\nu(dx)$, that is equivalent to $\int_X[\chi_e(z(x))-\chi_e(-z(x))]\nu(dx)=0$ or $\hat{\nu}(z)\in\mathbf{R}$. If $0<l<1$, then $\mu([x\in X:\hat{\nu}(x)\le l])=\mu([x:1-\hat{\nu}(x)\ge 1-l])\le\int_X(1-\hat{\nu}(x))\mu(dx)/(1-l)=\int_X(1-\hat{\mu}(x))\nu(dx)/(1-l)$ due to the Fubini theorem.

(b). Let $\nu(dx)=\gamma(x)m(dx)$, where $\gamma(x)=C\times\exp(-|x|^2)$, $C>0$, $\nu(X)=1$. Then $F(\gamma)(z)=:\hat{\gamma}(z)\ge 0$, and $\hat{\gamma}(0)=1$ and γ is the continuous positive definite function with $\gamma(z)\to 0$ whilst $|z|\to\infty$. In view of (a) we deduce that

$$\mu([x:\|x\|\ge tR])\le\int_X[1-\hat{\mu}(y\xi)]\nu(dy)/(1-l),$$

where $|\xi|=1/t$, $t>0$, $l=l(R)$. Estimating integrals, we get (b).

2.8. Lemma. *Let in the notation of Proposition* 2.7 $\nu_\xi(dx)=\gamma_\xi(x)m(dx)$, $\gamma_\xi(x)=C(\xi)\exp(-|x\xi|^2)$, $\nu_\xi(X)=1$, $\xi\ne 0$, *then a measure ν_ξ is weakly converging to the Dirac measure δ_0 with the support in $0\in X$ for $|\xi|\to\infty$.*

Proof. We have:

$$C(\xi)^{-1}=C_q(\xi)^{-1}=\sum_{l\in\mathbf{Z}}[p^{lq}-p^{(l-1)q}]\exp(-p^{2l}|\xi|^2)<\infty,$$

where the sum by $l<0$ does not exceed 1, $q=jn$, $j=dim_{\mathbf{K}}X$, $n=dim_{\mathbf{Q_p}}\mathbf{K}$. Here $\mathbf{K}$ is considered as the Banach space $\mathbf{Q_p^n}$ with the following norm $|*|_p$ equivalent to $|*|_{\mathbf{K}}$, for

$x = (x_1, \dots, x_j) \in X$ with $x_l \in \mathbf{K}$ as usually $|x|_p = \max_{1 \le l \le j} |x_l|_p$, for $y = (y_1, \dots, y_n) \in \mathbf{K}$ with $y_l \in \mathbf{Q_p}$: $|y|_p := \max_{1 \le l \le n} |y_l|_{\mathbf{Q_p}}$. Further we infer that

$$p^{l+s} \sum_{x_l \ne 0} \exp\left(2\pi i \sum_{i=l}^{-s-1} x_i p^{i+s}\right) = \int_{p^{l+s}}^{1} \exp(2\pi i \phi) d\phi + \beta(s),$$

where $s + l < 0$, $\lim_{s \to -\infty} (\beta(s) p^{-s-l}) = 0$, therefore, $sup[|\hat{\gamma}_1(z)|_{\mathbf{R}} |z|_X : z \in X, \, |z| \ge p^3] \le 2$. Then taking $0 \ne \xi \in \mathbf{K}$ and carrying out the substitution of variable for continuous and bounded functions $f : X \to \mathbf{R}$ we get

$$\lim_{|\xi| \to \infty} \int_X f(x) \nu_\xi(dx) = f(0).$$

This means that ν_ξ is weakly converging to δ_0 for $|\xi| \to \infty$.

2.9. Theorem. *Let μ_1 and μ_2 be two measures in $\mathsf{M}(X)$ such that $\hat{\mu}_1(f) = \hat{\mu}_2(f)$ for each $f \in \Gamma$, where $X = c_0(\alpha, K)$, $\alpha \le \omega_0$, Γ is a vector subspace in a space of continuous functions $f : X \to \mathbf{K}$ separating points in X. Then $\mu_1 = \mu_2$.*

Proof. Remind that a measure defined on the Borel σ-algebra of a topological space is called a Borel measure. The Borel measure μ in the topological space X is called τ-smooth, if for each increasing net $\{U_j : j \in J\}$ of open subsets U_j in X with a directed set J the equality $\mu(\bigcup_{j \in J} U_j) = \lim_{j \in J} \mu(U_j)$ is satisfied. In the set J a relation $\le$ directs J if it satisfies conditions $(D1 - D3)$:

$(D1)$ if $x \le y$ and $y \le z$ in J, then $x \le z$;

$(D2)$ $x \le x$ for each $x \in J$;

$(D3)$ for all $x, y \in J$ there exists an element $z \in J$ so that $x \le z$ and $y \le z$.

In the family $\mathcal{M}(X)$ of all Borel probability measures in the topological Tychonoff (completely regular) space X the weak topology is defined by the base of neighborhoods of a measure $\mu \in \mathcal{M}(X)$ of the form $U(\mu; f_1, \dots, f_k; \varepsilon) := \{\nu \in \mathcal{M}(X) : \quad |\int_X f_i d\nu - \int_X f_i d\mu| < \varepsilon, i = 1, \dots, k\}$, where $k \in \mathbf{N}$, $\varepsilon > 0$, $f_1, \dots, f_k$ are functions from the space $C_b(X, \mathbf{C})$ of all bounded continuous complex-valued functions on X.

Proposition 4.5 §I.4[VTC85] states that the set $\mathcal{M}_\tau(X)$ of all τ-smooth probability measures in a topological group X with the convolution operation of measures and the weak topology in $\mathcal{M}_\tau(X)$ is the topological semigroup with the neutral element δ_0. This semigroup $\mathcal{M}_\tau(X)$ is Abelian, if the group X is Abelian.

Let at first $\alpha < \omega_0$, then due to continuity of the convolution $\gamma_\xi * \mu_j$ by ξ, and Proposition 4.5 §I.4[VTC85] and Lemma 2.8 we get $\mu_1 = \mu_2$, since the family Γ generates $Bf(X)$. Now let $\alpha = \omega_0$, $A = \{x \in X : \; (f_1(x), \dots, f_n(x)) \in S\}$, ν_j be an image of a measure μ_j for a mapping $x \mapsto (f_1(x), \dots, f_n(x))$, where $S \in Bf(\mathbf{K^n})$, $f_j \in X \hookrightarrow X^*$. Then $\hat{\nu}_1(y) = \hat{\mu}_1(y_1 f_1 + \dots + y_n f_n) = \hat{\mu}_2(y_1 f_1 + \dots + y_n f_n) = \hat{\nu}_2(y)$ for each $y = (y_1, \dots, y_n) \in \mathbf{K^n}$, consequently, $\nu_1 = \nu_2$ on E.

Let X be a set and Γ be a subset in the family $\mathbf{R}^X$ of all real-valued mappings from X. The set $C_{f_1, \dots, f_k; B} := \{x \in X : (f_1(x), \dots, f_k(x)) \in B\}$ with $B \in Bf(\mathbf{R}^k)$ is called the cylindrical set relative to the pair (X, Γ). The Borel set B is called the base of the cylinder and $f_1, \dots, f_k$ are called its generator elements. The set of all cylinders is denoted by $\mathcal{C}(X, \Gamma)$.

Remind that a finite Borel measure μ on a Hausdorff topological space X is called Radon, if $\mu(B) = \sup\{\mu(K) : \quad K \subset B, K \text{ is compact}\}$ for each $B \in Bf(X)$. If the latter condition is satisfied for the set $B = X$, then the measure μ is called tight.

The Prohorof's theorem 3.4 §1.3 [VTC85] states that if X is a completely regular Hausdorff topological space, Γ is some family of real-valued continuous functions from X separating points in X, if also a finitely additive function $\mu : \mathcal{C}(X,\Gamma) \to [0,1]$ with $\mu(X) = 1$ is regular and tight, then there exists a unique extension of μ up to a Radon probability measure on X.

Further we can use the Prohorov theorem 3.4 §1.3 [VTC85], since compositions of $f \in \Gamma$ with continuous functions $g : \mathbf{K} \to \mathbf{R}$ generate a family of real-valued functions separating points of X. This finishes the proof.

2.10. Proposition. *Let μ_l and μ be measures in $\mathsf{M}(X_l)$ and $\mathsf{M}(X)$ respectively, where $X_l = c_0(\alpha_l, \mathbf{K})$, $\alpha_l \le \omega_0$, $X = \prod_1^n X_l$, $n \in \mathbf{N}$. Then the condition $\hat{\mu}(z_1,\dots,z_n) = \prod_{l=1}^n \hat{\mu}_l(z_l)$ for each $(z_1,\dots,z_n) \in X \hookrightarrow X^*$ is equivalent to $\mu = \prod_{l=1}^n \mu_l$.*

Proof. Let $\mu = \prod_{l=1}^n \mu_l$, then $\hat{\mu}(z_1,\dots,z_n) = \int_X \chi_e(\sum z_l(x_l)) \prod_{l=1}^n \mu_l(dx_l) = \prod_{l=1}^n \int_{X_l} \chi_e(z_l(x_l))\mu_l(dx_l)$. The reverse statement follows from Theorem 2.9.

2.11. Proposition. *Let X be a Banach space over* $\mathbf{K}$*; suppose μ, μ_1 and μ_2 are probability measures on X. Then the following conditions are equivalent:*

μ is the convolution of two measures μ_j, $\mu = \mu_1 * \mu_2$, and

$\hat{\mu}(z) = \hat{\mu}_1(z)\hat{\mu}_2(z)$ for each $z \in X$.

Proof. Let $\mu = \mu_1 * \mu_2$. This means by the definition that μ is the image of the measure $\mu_1 \otimes \mu_2$ for the mapping $(x_1,x_2) \mapsto x_1 + x_2$, $x_j \in X$, consequently,

$$\hat{\mu}(z) = \int_{X\times X} \chi_e(z(x_1+x_2))(\mu_1\otimes\mu_2)(d(x_1,x_2))$$

$$= \prod_{l=1}^{2} \int_X \chi_e(z(x_l))\mu_l(dx_l) = \hat{\mu}_1(z)\hat{\mu}_2(z).$$

On the other hand, if $\hat{\mu}_1\hat{\mu}_2 = \hat{\mu}$, then $\hat{\mu} = (\mu_1 * \mu_2)^{\wedge}$ and due to Theorem 2.9 above for real measures we have $\mu = \mu_1 * \mu_2$.

2.12. Corollary. *Let ν be a probability measure on the Borel σ-algebra $Bf(X)$ and $\mu * \nu = \mu$ for each μ, then$\nu = \delta_0$.*

Proof. If $z_0 \in X \hookrightarrow X^*$ and $\hat{\mu}(z_0) \ne 0$, then from $\hat{\mu}(z_0)\hat{\nu}(z_0) = \hat{\mu}(z_0)$ it follows that $\hat{\nu}_0(z_0) = 1$. From the property 2.6(6) we get that there exists $m \in \mathbf{N}$ with $\hat{\mu}(z) \ne 0$ for each z with $\|z\| = p^{-m}$, since $\hat{\mu}(0) = 1$. Then $\hat{\nu}(z+z_0) = 1$, that is, $\hat{\nu}|_{(B(X,z_0,p^{-m}))} = 1$. Since μ are arbitrary we get $\hat{\nu}|_X = 1$, so we infer that $\nu = \delta_0$ due to §2.6 and §2.9.

2.13. Corollary. *Let X and Y be Banach spaces over* $\mathbf{K}$,

(a) let μ and ν be probability measures on X and Y respectively, suppose $T : X \to Y$ is a continuous linear operator. A measure ν is an image of μ for T if and only if $\hat{\nu} = \hat{\mu} \circ T^*$, where $T^* : Y^* \to X^*$ is an adjoint operator. (b). A characteristic functional of a real measure μ on $Bf(X)$ is real if and only if μ is symmetric.

Proof. It follows immediately from §2.6 and §2.9.

2.14. Definition. We say that a real probability measure μ on $Bf(X)$ for a Banach space X over $\mathbf{K}$ and $0 < q < \infty$ has a weak q-th order if $\psi_{q,\mu}(z) = \int_X |z(x)|^q \mu(dx) < \infty$ for each $z \in X^*$. The weakest vector topology in X^* relative to which all $(\psi_{q,\mu} : \mu)$ are continuous is denoted by $\tau_q := \tau_q(X^*,X)$.

2.15. Theorem. *A characteristic functional $\hat{\mu}$ of a real probability Radon measure μ on $Bf(X)$ is continuous in the topology τ_q for each $q > 0$.*

Proof. For each $c > 0$ there exists a compact subset $S_c := S \subset X$ such that $\mu(S) > 1 - c/4$ and

$$|1-\hat{\mu}(z)| \le |\int_S (1-\chi_e(z(x)))\mu(dx)| + |\int_{X\setminus S}(1-\chi_e(z(x)))\mu(dx)| \le |1-\hat{\mu}_c(z)| + c/2,$$

where $\mu_c(A) = (\mu(A\cap S)/\mu(S)$ and $A \in Bf(X)$. Define the measure $\mu_c := \mu(B\cap S_c)/\mu(S_c)$ for each $B \in Bf(X)$. Then the measure μ_c has the compact support, consequently, it has any strong order p, $0 < p < \infty$, that is by the definition $\int_X \|x\|^p d\mu_c(x) < \infty$. Since $\tau_{p_1}(X^*,X) \subset \tau_{p_2}(X^*,X)$ for each $0 < p_1 < p_2 < \infty$, then we can consider $0 < p \le 1$ without loss of generality. In view of the inequality $|1-\exp(it)| \le 2|t|^p$ for each $t \in \mathbf{R}$ we deduce that $|1-\hat{\mu}_c(x^*)| \le 2\psi_{p,\mu_c}(x^*)$ for every $x^* \in X^*$, where $i = (-1)^{1/2}$. Thus $|1-\hat{\mu}(x^*)| \le 2\psi_{p,\mu_c}(x^*) + c/2$ for each $x^* \in X^*$. Therefore, if $\psi_{p,\mu_c}(x^*) < c/4$, then $|1-\hat{\mu}(x^*)| < c$, consequently, the characteristic functional $\hat{\mu}$ is continuous at zero in the topology $\tau_p(X^*,X)$.

2.16. Proposition. *For a completely regular space X with zero small inductive dimension $ind(X) = 0$ the following statements are accomplished:*

(a). if (μ_β) is a bounded net of measures in $\mathsf{M}(X)$ that weakly converges to a measure μ in $\mathsf{M}(X)$, then $(\hat{\mu}_\beta(f))$ converges to $\hat{\mu}(f)$ for each continuous $f : X \to \mathbf{K}$; if X is separable and metrizable then $(\hat{\mu}_\beta)$ converges to $\hat{\mu}$ uniformly on subsets that are uniformly equicontinuous in $C(X,\mathbf{K})$;

(b). if M is a bounded dense family in a ball of the space $\mathsf{M}(X)$ for measures in $\mathsf{M}(X)$, then a family $(\hat{\mu} : \mu \in M)$ is equicontinuous on a locally $\mathbf{K}$-convex space $C(X,\mathbf{K})$ in a topology of uniform convergence on compact subsets $S \subset X$.

Proof. (a). Functions $\chi_e(f(x))$ are continuous and bounded on X, where $\hat{\mu}(f) = \int_X \chi_e(f(x))\mu(dx)$.

The Ranga Rao's proposition 1.3.9[VTC85] states that if a net $\{\mu_j\}$ of Borel probability measures in a separable metric space X weakly converges to a Borel probability measure μ in X and if $\Gamma \subset C_b(X,\mathbf{C})$ is a bounded pointwise equicontinuous subset, then $\lim_j \sup_{f\in\Gamma} |\int_X f d\mu_j - \int_X f d\mu| = 0$.

Then Statement (a) follows from the definition of the weak convergence and Proposition 1.3.9[VTC85], since the linear span $span_{\mathbf{C}}\{\chi_e(f(x)) : f \in C(X,\mathbf{K})\}$ over the complex field is dense in $C(X,\mathbf{C})$.

(b). For each $c > 0$ there exists a compact subset $S_c := S \subset X$ such that $|\mu|(S) > |\mu(X)| - c/4$. The set $V_c := \{f \in C(X,\mathbf{K}) : |f(x)| < c^{1/2}\ \forall x \in S_c\}$ is a neighborhood of zero in the topology τ_c in $C(X,\mathbf{K})$ of the uniform convergence on compact subsets in X. Without loss of generality we can consider $\mu(X) = 1$ renormalizing μ in case of necessity. Therefore, for $\mu \in \mathsf{M}$ and $f \in C(X,\mathbf{K})$ with $|f(x)|_{\mathbf{K}} < c < 1$ for $x \in S$ we get

$$|\mu(X) - Re(\hat{\mu}(f)| = 2|\int_X [(\chi_e^{1/2}(f(x)) - \chi_e^{1/2}(-f(x))/(2i)]^2\mu(dx)|$$

$$\le 2|\int_S [(\chi_e^{1/2}(f(x)) - \chi_e^{1/2}(-f(x))/(2i)]^2\mu(dx)|$$

$$+2|\int_{X\setminus S}[(\chi_e^{1/2}(f(x)) - \chi_e^{1/2}(-f(x))/(2i)]^2\mu(dx)| < c.$$

We have that X is the T_1-space and for each point x and each closed subset S in X with $x \notin S$ there is a continuous function $h : X \to B(\mathbf{K},0,1)$ such that $h(x) = 0$ and $h(S) = \{1\}$. Thus the family $\{Re\hat{\mu} : \mu \in \mathsf{M}(X)\}$ is equicontinuous at zero in $C(X,\mathbf{R})$ in the topology τ_c.

2.17. Theorem. *Let X be a Banach space over* $\mathbf{K}$, $\gamma : \Gamma \to \mathbf{C}$ *be a continuous positive definite function,* (μ_β) *be a bounded weakly relatively compact net in the space* $\mathsf{M}_t(X)$ *of Radon norm-bounded measures and there exists* $\lim_\beta \hat{\mu}_\beta(f) = \gamma(f)$ *for each* $f \in \Gamma$ *and uniformly on compact subsets of the completion* $\tilde{\Gamma}$, *where* $\Gamma \subset C(X, \mathbf{K})$ *is a vector subspace separating points in X. Then* (μ_β) *weakly converges to* $\mu \in \mathsf{M}_t(X)$ *with* $\hat{\mu}|_\Gamma = \gamma$.

Proof. We show that μ_β has a unique limit point in the space $\mathcal{M}_t(X)$ of Radon probability measures in X. Suppose the contrary that there exist two distinct limit points $\mu_1, \mu_2 \in \mathcal{M}_t(X)$. Then there would exist two nets $\mu_{1,\rho}$ and $\mu_{2,\phi}$ weakly converging to μ. That is, $\lim_\rho \hat{\mu}_{1,\rho} = \lim_\phi \hat{\mu}_{2,\phi} = \lim_\beta \hat{\mu}_\beta(f) = \gamma(f)$, $f \in \Gamma$. On the other hand, from the weak convergence it follows that $\lim_\rho \hat{\mu}_{1,\rho}(f) = \hat{\mu}_1(f)$ and $\lim_\phi \hat{\mu}_{2,\phi}(f) = \hat{\mu}_2(f)$ for each $f \in \Gamma$. From the letter equalities we deduce that $\hat{\mu}_2 = \hat{\mu}_1 = \gamma$. But in view of Theorem 2.9 we get $\mu_1 = \mu_2$. Thus $\{\mu_\beta\}$ has a unique limit point $\mu \in \mathcal{M}_t(X)$. From the relative compactness of $\{\mu_\beta\}$ it follows that μ_β weakly converges to μ and $\hat{\mu}(f) = \gamma(f)$ for every $f \in \Gamma$.

2.18. Theorem. (a). *A bounded family M of measures in* $\mathsf{M}(\mathbf{K}^\mathbf{n})$ *is weakly relatively compact if and only if a family* $(\hat{\mu} : \mu \in M)$ *is equicontinuous on* $\mathbf{K}^\mathbf{n}$.

(b). *If* $(\mu_j : j \in \mathbf{N})$ *is a bounded sequence of measures in* $\mathsf{M}_t(\mathbf{K}^\mathbf{n})$, $\gamma : \mathbf{K}^\mathbf{n} \to \mathbf{C}$ *is a continuous and positive definite function,* $\hat{\mu}_j(y) \to \gamma(y)$ *for each* $y \in \mathbf{K}^\mathbf{n}$, *then* (μ_j) *weakly converges to a measure* μ *with* $\hat{\mu} = \gamma$.

(c). *A bounded sequence of measures* (μ_j) *in* $\mathsf{M}_t(\mathbf{K}^\mathbf{n})$ *weakly convereges to a measure* μ *in* $\mathsf{M}_t(\mathbf{K}^\mathbf{n})$ *if and only if for each* $y \in \mathbf{K}^\mathbf{n}$ *there exists* $\lim_{j\to\infty} \hat{\mu}_j(y) = \hat{\mu}(y)$.

(d). *If a bounded net* $\{\mu_\beta : \beta \in J\}$ *in* $\mathsf{M}_t(\mathbf{K}^\mathbf{n})$ *converges uniformly on each bounded subset in* $\mathbf{K}^\mathbf{n}$, *then* (μ_β) *converges weakly to a measure* μ *in* $\mathsf{M}_t(\mathbf{K}^\mathbf{n})$, *where* $n \in \mathbf{N}$.

Proof. (a). The relative compactness of the family M implies due to the Prohorov's theorem 1.3.6[VTC85] and Proposition 2.16 above that the family $\{\hat{\mu} : \mu \in M\}$ is equicontinuous. Vise versa if the family $\{\hat{\mu} : \mu \in M\}$ is equicontinuous then due to Proposition 2.7 the family M is dense, consequently, it is weakly relatively compact.

(b). We have the following inequality: $\lim_m \sup_{j>m} \mu_j([x \in \mathbf{K}^\mathbf{n} : |x| \ge tR]) \le 2 \int_{\mathbf{K}^\mathbf{n}} (1 - Re(\eta(\xi y)))\nu(dy)$ with $|\xi| = 1/t$ due to §2.7 and §2.8. In view of Theorem 2.17 (μ_j) converges weakly to μ with $\hat{\mu} = \gamma$.

(c). If μ_k weakly converges to μ, then $\hat{\mu}_k(y)$ converges to $\hat{\mu}(y)$ for each vector $y \in \mathbf{K}^\mathbf{n}$. The converse statement follows from (b) and the bijective correspondence between measures and characteristic functionals.

(d). From the condition it follows that the function $\gamma := \lim \hat{\mu}_\beta$ is positive definite. The uniform convergence of $\hat{\mu}_\beta$ to χ in some neighborhood of zero implies the continuity of γ in this neighborhood. Due to the positive definiteness of γ this in its turn implies that γ is continuous everywhere. The space $\mathbf{K}^\mathbf{n}$ is the countable union of bounded subsets, for example, of balls. Therefore, there exists a subsequence $\{\beta_k : k \in \mathbf{N}\}$ such that $\lim_k \hat{\mu}_{\beta_k}(y) = \gamma(y)$ for each $y \in \mathbf{K}^\mathbf{n}$. In view of (b) we have that μ_{β_k} weakly converges to μ, where μ is a measure with $\hat{\mu} = \gamma$.

Suppose now that μ_β does not converge to μ relative to the weak topology. Consider a metric ρ the convergence relative to which is equivalent to the weak convergence of measures. This is possible, since the space $C_b(X, \mathbf{C})$ is separable. In view of the Nagata-Smirnov's metrization theorem 4.4.7 [Eng86] a topological space is metrizable if and only if it is regular and has a σ-locally finite base. Recall that a family of subsets is called σ-locally finite if it can be presented as a countable union of locally finite families.

Then $\rho(\mu_\beta,\mu)$ does not converge to zero, hence there exists $c>0$ so that the set $J_0 := \{\beta\in J:\ \rho(\mu_\beta,\mu)\geq c\}$ is co-final with J. The net $\{\hat{\mu}_\beta : \beta\in J_0\}$ converges to γ uniformly on each bounded subset. In accordance with the fact proved above there exists the sequence $\{\beta_k : k\}\subset J_0$ so that μ_{β_k} weakly converges to μ. But this is impossible, since $\rho(\mu_{\beta_k},\mu)\geq\varepsilon$. Thus $\lim_{\beta\in J}\rho(\mu_\beta,\mu)=0$ and inevitably μ_β weakly converges to μ.

2.19. Corollary. *If $(\hat{\mu}_\beta)\to 1$ uniformly on some neighborhood of 0 in $\mathbf{K^n}$ for a bounded net of measures μ_β in $\mathsf{M}_t(\mathbf{K^n})$, then (μ_β) converges weakly to δ_0.*

Proof. Since $\hat{\mu}_\beta$ is positive definite for each β, then $|1-\hat{\mu}_\beta(2y)| = |1-\hat{\mu}_\beta(y)+\hat{\mu}_\beta(y)-\hat{\mu}_\beta(2y)|\leq|1-\hat{\mu}_\beta(y)|+[2(1-Re\hat{\mu}_\beta(y))]^{1/2}$. From this it follows that the convergence of the net $\hat{\mu}_\beta$ to 1 is uniform on each bounded subset in $\mathbf{K^n}$. Applying Statement (d) of the preceding theorem we deduce the demonstration of this corollary.

2.20. Definition. A family of probability measures $M\subset\mathsf{M}_t(X)$ for a Banach space X over $\mathbf{K}$ is called planely concentrated if for each $c>0$ there exists a $\mathbf{K}$-linear subspace $S\subset X$ with $dim_{\mathbf{K}}S=n<\aleph_0$ such that $\inf(\mu(S^c):\mu\in M)>1-c$, where $S^c := \{x\in X:\ \inf_{y\in S}\|y-x\|\leq c\}$. The Banach space $\mathsf{M}_t(X)$ is supplied with the following norm $\|\mu\| := |\mu|(X)$.

2.21. Lemma. *Let S and X be the same as in §2.20; $z_1,\dots,z_m\in X^*$ be a separating family of points in S. Then a set $E := S^c\cap(x\in X : |z_j(x)|\leq r_j;\ j=1,\dots,m)$ is bounded for each $c>0$ and $r_1,\dots,r_m\in(0,\infty)$.*

Proof. A space S is isomorphic with $\mathbf{K^n}$, consequently, $p(x)=\max(|z_j| : j=1,\dots,m)$ is the norm in S equivalent to the initial norm. Suppose that in E there exists a sequence $\{x_k\}$ so that $\lim_k\|x_k\|=\infty$. Then for some $1\leq j\leq n$ we would have $\lim_k|z_j(x_k)|=\infty$ contradicting $x_k\in E$ for each $k\in\mathbf{N}$. Thus E is contained in the ball $B(X,0,R)$ for some $0<R<\infty$.

2.22. Theorem. *Let X be a Banach space over $\mathbf{K}$ with a family $\Gamma\subset X^*$ separating points in a family of probability measures $M\subset\mathsf{M}_t(X)$. Then M is weakly relatively compact if and only if a family $\{\mu_z : \mu\in M\}$ is weakly relatively compact for each $z\in\Gamma$ and M is planely concentrated, where μ_z is an image measure on $\mathbf{K}$ of a measure μ induced by z.*

Proof. Remind the following. Suppose that X is a metric space with a metric ρ in it. On the space $\mathcal{M}(X)$ introduce the Prohorov's metric $d(\mu,\nu) := \inf\{b>0 : \mu(A)\leq\nu(A^b)+b\ \forall A\in Bf(X)\}$, where $A^b := \{x\in X : \rho(x,A)<b\}$, $b>0$. The Prohorov's criteria states that if a net $\{\mu_j : j\in J\}$ of probability measures in the metric space X converges in the metric d to a probability measure μ, then μ_j weakly converges to μ. If $\mu\in\mathcal{M}_\tau(X)$, then from the weak convergence of μ_j to μ it follows that $\lim_j d(\mu_j,\mu)=0$. Thus the weak topology in $\mathcal{M}_\tau(X)$ is metrizable.

The necessity follows from Lemmas 2.5, 2.21 and the Prohorov's criteria.

Prove the sufficiency. For each $c>0$ and $n\in\mathbf{N}$ find a finite-dimensional subspace $S_{n,c}\subset X$ so that for each $\mu\in M$ we have $\mu(Y_{n,c})>1-c2^{-n-1}$, where $Y_{n,c} := cl(S_{n,c}^{c2^{-n-1}})$ is the closure of the $c2^{-n-1}$ open neighborhood of the set $S_{n,c}$. Consider now a finite set of functionals $x_1^*,\dots,x_{k_n}^*\in\Gamma$ separating points in $S_{n,c}$. Choose real numbers $r_1,\dots,r_{k_n}\in(0,\infty)$ satisfying the condition:

$\inf_{\mu\in M}\mu_{x_j^*}(B(\mathbf{K},0,r_j))>1-c2^{-n-1}/k_n$ for each $j=1,\dots,k_n$, where $\mu_{x_j^*}(A) := \mu(\{x\in X:\ x_j^*(x)\in A\})$ for each Borel subset A in the field $\mathbf{K}$..

In accordance with Lemma 2.21 the set $K := \bigcap_{n=1}^\infty(Y_{n,c}\cap\{x : |x_j^*(x)|<r_j\ \ \forall j=1,\dots,k_n\})$ is bounded. The set K is closed and $K\subset S_{n,c}^{c2^{-n-1}}$ for each $c>0$ and $n\in\mathbf{N}$.

In view of Lemma 2.5 the set K is compact. Moreover,

$$\mu(X\setminus K)\le\sum_{n=1}^{\infty}(\mu(X\setminus Y_{n,c})+\sum_{j=1}^{k_n}\mu_{x_j^*}(\mathbf{K}\setminus B([\mathbf{K},0,r_j))\le c.$$

Applying the Prohorov's criteria we deduce that the family M is weakly relatively compact.

2.23. Theorem. *For X and Γ the same as in Theorem 2.22 a sequence $\{\mu_j : j\in\mathbf{N}\}\subset \mathsf{M}_t(X)$ is weakly convergent to $\mu\in\mathsf{M}_t(X)$ if and only if two conditions are satisfied*

(a) for each $z\in\Gamma$ there exists $\lim_{j\to\infty}\hat{\mu}_j(z)=\hat{\mu}(z)$ and

(b) a family $\{\mu_j\}$ is planely concentrated.

Proof. The necessity of these two conditions follows from Theorem 22. Now prove the sufficiency. In view of Theorem 2.18 and Condition (a) we have that $(\mu_k)_{x^*}$ weakly converges in $\mathbf{K}$ for each $x^*\in\Gamma$. Due to Theorem 22 the sequence $\{\mu_k : k\}$ is weakly relatively compact. Applying Theorem 17 we deduce the statement of this theorem.

2.24. Proposition. *Let X be a completely regular space with the zero small inductive dimension $ind(X)=0$, $\Gamma\subset C(X,\mathbf{K})$ be a vector subspace separating points in X, $(\mu_n : n\in\mathbf{N})\subset\mathsf{M}_t(X)$, $\mu\in\mathsf{M}_t(X)$, $\lim_{n\to\infty}\hat{\mu}_n(f)=\hat{\mu}(f)$ for each $f\in\Gamma$. Then (μ_n) is weakly convergent to μ relative to the weakest topology $\sigma(X,\Gamma)$ in X relative to which all $f\in\Gamma$ are continuous.*

Proof. Recall the A.D. Alexandroff's theorem. Let $\{\mu_j : j\in J\}$ be a net of measures in $\mathcal{M}(X)$, $\mu\in\mathcal{M}_\tau(X)$. If X is a metric space, then the τ-smoothness of μ is not necessary. Then the following conditions are equivalent:

(a) μ_j weakly converges to μ;

(b) $\overline{\lim}_j\mu_j(F)\le\mu(F)$ for each closed subset F in X;

(c) $\underline{\lim}_j\mu_j(U)\ge\mu(U)$ for each open U in X;

(d) $\lim_j\mu_j(B)=\mu(B)$ for each $B\in Bf(X)$ so that $\mu(Fr(B))=0$.

In view of A.D. Alexandroff's theorem it is sufficient to show that $\underline{\lim}_j\mu_j(U)\ge\mu(U)$ for each $\sigma(X,\Gamma)$ open U in X. At first show this relation for each open $U\in\mathcal{C}(X,\Gamma)$. Consider an arbitrary cylinder $U_0:=\{x\in X : (f_1(x),\dots,f_n(x))\in V_n\}$, where $f_1,\dots,f_n\in\Gamma$, V_n is open in $\mathbf{K^n}$. Under the mapping $x\mapsto(f_1(x),\dots,f_n(x))$ we get the images ν, ν_n in $\mathbf{K^n}$ of measures μ, μ_n, where $n\in\mathbf{N}$. For any $y=(y_1,\dots,y_n)\in\mathbf{K^n}$ we have

$$\lim_n\hat{\nu}_n(y)=\lim_n\hat{\mu}\left(\sum_{k=1}^n y_kf_k\right)=\hat{\mu}\left(\sum_{k=1}^n y_kf_k\right)=\hat{\nu}(y).$$

In view of Theorem 18 ν_n weakly converges to ν, consequently, $\underline{\lim}_j\nu_j(V_n)\ge\nu(V_n)$ and inevitably $\underline{\lim}_j\mu_j(U_0)\ge\mu(U_0)$.

Let now U be an arbitrary open subset and $U_0\subset U$. Then $\underline{\lim}_j\mu_j(U)\ge\underline{\lim}_j\mu_j(U_0)\ge\mu(U_0)$.

By the definition of the topology $\sigma(X,\Gamma)$ each $\sigma(X,\Gamma)$-open subset U in X is a union of some family of open cylindrical subsets. Denote by $\mathcal{U}$ the family of all open cylinders $U_0\subset U$. The family $\mathcal{U}$ is ordered by inclusion and the measure μ is τ-smooth, hence $\mu(U)=\sup_{U_0\in\mathcal{U}}\mu(U_0)$, consequently, $\underline{\lim}_j\mu_j(U)\ge\mu(U)$.

2.25. Let $(X,\mathsf{U})=\prod_\lambda(X_\lambda,\mathsf{U}_\lambda)$ be a product of measurable completely regular Radon spaces $(X_\lambda,\mathsf{U}_\lambda)=(X_\lambda,\mathsf{U}_\lambda,\mathsf{K}_\lambda)$, where K_λ are compact classes approximating from below each measure μ_λ on $(X_\lambda, \mathsf{U}_\lambda)$, that is, for each $c > 0$ and elements A of an algebra U_λ there is $S \in \mathsf{K}_\lambda$, $S \subset A$ with $\|A \setminus S\|_{\mu_\lambda} < c$.

2.26. Definition. Let X be a Banach space over $\mathbf{K}$, then a mapping $f : X \to \mathbf{C}$ is called pseudo-continuous, if its restriction $f|_L$ is uniformly continuous for each subspace $L \subset X$ with the finite dimension $dim_{\mathbf{K}} L < \aleph_0$ over the field $\mathbf{K}$.

Let Γ be a family of mappings $f : Y \to \mathbf{K}$ of a set Y into a field $\mathbf{K}$. We denote by $\hat{\mathsf{C}}(Y,\Gamma)$ the minimal σ-algebra (that is called cylindrical) generated by an algebra $\mathsf{C}(Y,\Gamma)$ of subsets of the form $C_{f_1,\dots,f_n;E} := \{x \in X : (f_1(x),\dots,f_n(x)) \in S\}$, where $S \in Bf(\mathbf{K^n})$, $f_j \in \Gamma$. We supply Y with a topology $\tau(Y)$ which is generated by a base $(C_{f_1,\dots,f_n;E} : f_j \in \Gamma, E$ is open in $\mathbf{K^n})$.

2.27. Theorem. Non-Archimedean analog of the Bochner-Kolmogorov theorem. *Let X be a Banach space over $\mathbf{K}$, X^a be its algebraically dual $\mathbf{K}$-linear space (that is, of all linear mappings $f : X \to \mathbf{K}$ not necessarily continuous). A mapping $\theta : X^a \to \mathbf{C}$ is a characteristic functional of a probability measure μ with values in $\mathbf{R}$ and is defined on $\hat{\mathsf{C}}(X^a, X)$ if and only if θ satisfies conditions 2.6(3,5) for $(X^a, \tau(X^a))$ and is pseudo-continuous on X^a.*

Proof. (I). For $dim_{\mathbf{K}} X = card(\alpha) < \aleph_0$ a space X^a is isomorphic with $\mathbf{K}^\alpha$, hence the statement of theorem for a measure μ follows from Theorems 2.9 and 2.18 above, since $\theta(0) = 1$ and $|\theta(z)| \le 1$ for each z.

(II). We consider now the case of μ with values in $\mathbf{R}$ and $\alpha < \omega_0$. In §2.6 (see also §2.16-18,24) it was proved that $\theta = \hat{\mu}$ has the desired properties for real probability measures μ. On the other hand, there is θ which satisfies the conditions of the theorem. Let $\theta_\xi(y) = \theta(y) h_\xi(y)$, where $h_\xi(y) = F[C(\xi)\exp(-\|x\xi\|^2)](y)$ (that is, the Fourier transform by x), $\nu_\xi(\mathbf{K}^\alpha) = 1$, $\nu_\xi(dx) = C(\xi)\exp(-\|x\xi\|^2) m(dx)$ (see Lemma 2.8), $\xi \ne 0$. Then $\theta_\xi(y)$ is positive definite and is uniformly continuous as a product of two such functions. Moreover, $\theta_\xi(y) \in L^1(\mathbf{K}^\alpha, m, \mathbf{C})$. For $\xi \ne 0$ a function $f_\xi(x) = \int_{\mathbf{K}^\alpha} \theta_\xi(y) \chi_e(x(y)) m(dy)$ is bounded and continuous, the function $\exp(-\|x\xi\|^2) =: s(x)$ is positive definite. Since ν_ξ is symmetric and weakly converges to δ_0, hence there exists $r > 0$ such that for each $|\xi| > r$ we have $\hat{\gamma}_\xi(y) = \int_{\mathbf{K}^\alpha} C(\xi)\exp(-\| \ x\xi\|_p^2)\chi_e(y(x)) m(dx)$ $= \int [\chi_e((y(x)) + \chi_e(-y(x))] 2^{-1} \exp(-\|x\xi\|_p^2) C(\ \xi) m(dx)/2 > 1 - 1/R$ for $|y| \le R$, consequently, $\hat{\gamma}_\xi(y) = \hat{\zeta}_\xi^2(y)$ for $|y| \le R$, where $\hat{\zeta}_\xi$ is positive definite uniformly continuous and has a uniformly continuous extension on $\mathbf{K}^\alpha$. Therefore, for each $c > 0$ there exists $r > 0$ such that $\|\nu_\xi - \kappa_\xi * \kappa_\xi\| < c$ for each $|\xi| > r$, where $\kappa_\xi(dx) = \zeta_\xi(x) m(dx)$ is a σ-additive non-negative measure.

Recall the following proposition about positive definite functions (see also IV.1.3 [VTC85]). Let $(\Omega, \mathcal{B})$ be a measurable space, ν be a σ-finite measure on $\mathcal{B}$ and let $f : \Omega \times \Omega \to \mathbf{C}$ be a measurable and $\nu \times \nu$-integrable positive definite function, then $\int_{\Omega \times \Omega} f d(\nu \times \nu) \ge 0$.

From this proposition we also use the following corollary. Let $(\Omega, \mathcal{B})$ be a measurable group, let also ν be a symmetric probability measure on $\mathcal{B}$ and let $g : \Omega \to \mathbf{C}$ be a measurable positive definite function, then $\int_\Omega g d(\nu * \nu) \ge 0$.

Hence due to this corollary there exists $r > 0$ such that $\int_{\mathbf{K}^\alpha} \theta_\xi(y) \chi_e(-x(y)) \nu_j(dy)$ ≥ 0 for each $|j| > r$, consequently, $f_\xi(x) = \lim_{|j| \to \infty} \int_{\mathbf{K}^\alpha} \theta_\xi(y)\ \chi_e(-x(y)) \nu_j(dy) \ge 0$. From the equality $F[F(\gamma_\xi)(-y)](x) = \gamma_\xi(x)$ and the Fubini theorem it follows that

$$\int f_\xi \chi_e(y(x)) h_j(x) m(dx) = \int \theta_\xi(u+y) \nu_j(du).$$

For $y = 0$ we get

$$\lim_{|\xi|\to\infty} \int f_\xi(x) m(dx) = \int f(x) m(dx) = \lim_{|\xi|\to\infty} \lim_{|j|\to\infty} \int f_\xi(x) h_j(x) m(dx)$$

$$\text{and } \lim_{|\xi|\to\infty} \lim_{|j|\to\infty} |\int_{\mathbf{K}^\alpha} \theta_\xi(u) \nu_j(du)| \le 1.$$

From Lemma 2.8 it follows that $\hat{f}(y) = \theta(y)$, since by Theorem 2.18 $\theta = \lim_{|\xi|\to\infty} \theta_\xi$ is a characteristic function of a probability measure on $Bf(\mathbf{K}^\alpha)$, where $f(x) = \int_{\mathbf{K}^\alpha} \theta(y) \chi_e(-x(y)) m(dy)$.

(III). Now let $\alpha = \omega_0$. It remains to show that the conditions imposed on θ are sufficient, because their necessity follows from the modification of 2.6 (since X has an algebraic embedding into X^a). The space X^a is isomorphic with $\mathbf{K}^\Lambda$ which is the space of all $\mathbf{K}$-valued functions defined on the Hamel basis Λ in X. The Hamel basis exists due to the Kuratowski-Zorn lemma (that is, each finite system of vectors in Λ is linearly independent over $\mathbf{K}$, each vector in X is a finite linear combination over $\mathbf{K}$ of elements from Λ). Let J be a family of all non-void subsets in Λ. For each $A \in J$ there exists a functional $\theta_A : \mathbf{K}^A \to \mathbf{C}$ such that $\theta_A(t) = \theta(\sum_{y\in A} t(y)y)$ for $t \in \mathbf{K}^A$. From the conditions imposed on θ it follows that $\theta_A(0) = 1$, θ_A is uniformly continuous and bounded on $\mathbf{K}^A$, moreover, it is positive definite (or due to 2.6(6) for each $c > 0$ there are n and $q > 0$ such that for each $j > n$ and $z \in \mathbf{K}^A$ the following inequality is satisfied:

$$(i)\ |\theta_A(z) - \theta_j(z)| \le cbq,$$

moreover, $L_j \supset \mathbf{K}^A$, q is independent on j, c and b. From (I,II) it follows that on $Bf(\mathbf{K}^A)$ there exists a probability measure μ_A such that $\hat{\mu}_A = \theta_A$. The family of measures $\{\mu_A : A \in J\}$ is consistent and bounded, since $\mu_A = \mu_E \circ (P^A_E)^{-1}$, if $A \subset E$, where $P^A_E : \mathbf{K}^E \to \mathbf{K}^A$ are the natural projectors. Indeed, in the case of measures with values in $\mathbf{R}$ each μ_A is the probability measure.

Remind the Kolmogorov's theorem (see also Theorem 1.1.4 [DF91]). Suppose that $(X, \mathcal{U}) = \prod_{j\in\Lambda}(X_j, \mathcal{U}_j)$ be the product of measurable Radon spaces $(X_j, \mathcal{U}_j) = (X_j, \mathcal{U}_j, \mathcal{K}_j)$, then each bounded quasi-measure on $(X, \mathcal{U})$ is a measure.

In view of the Kolmogorov's theorem on a cylindrical σ-algebra of the space $\mathbf{K}^\Lambda$ there exists the unique measure μ such that $\mu_A = \mu \circ (P^A)^{-1}$ for each $A \in J$, where $P^A : \mathbf{K}^\Lambda \to \mathbf{K}^A$ are the natural projectors. From $X^a = \mathbf{K}^\Lambda$ it follows that μ is defined on $\hat{C}(X^a, X)$. For μ on $\hat{C}(X^a, X)$ there exists its extension on $Af(X, \mu)$ (see §2.1).

2.28. Definition. [Sch89] A continuous linear operator $T : X \to Y$ for Banach spaces X and Y over $\mathbf{K}$ is called compact, if $T(B(X,0,1)) =: S$ is a compactoid, that is, for each neighbourhood $U \ni 0$ in Y there exists a finite subset $A \subset Y$ such that $S \subset U + co(A)$, where $co(A)$ is the least $\mathbf{K}$-absolutely convex subset in V containing A (that is, for each a and $b \in \mathbf{K}$ with $|a| \le 1$, $|b| \le 1$ and for each $x, y \in V$ the following inclusion $ax + by \in V$ is accomplished).

2.29. Let B_+ be a subset of non-negative functions which are $Bf(X)$-measurable and let C_+ be its subset of non-negative cylindrical functions. By $\hat{\mathsf{B}}_+$ we denote a family of functions $f \in \mathsf{B}_+$ such that $f(x) = \lim_n g_n(x)$, $g_n \in \mathsf{C}_+$, $g_n \geq f$. For $f \in \hat{\mathsf{B}}_+$ let

$$\int_X f(x)\mu_*(dx) := \inf_{g \geq f,\ g \in \mathsf{C}_+} \int_X g(x)\mu_*(dx).$$

2.30. Lemma. *A sequence of weak distributions (μ_{L_n}) of probability Radon measures is generated by a real probability measure μ on $Bf(X)$ of a Banach space X over* $\mathbf{K}$ *if and only if there exists*

$$(i)\ \lim_{|\xi| \to \infty} \int_X G_\xi(x)\mu_*(dx) = 1,$$

where $\int_X G_\xi(x)\mu_*(dx) := S_\xi(\{\mu_{L_n} : n\})$ and

$$S_\xi(\{\mu_{L_n}\}) := \lim_{n\to\infty} \int_{L_n} F_n(\gamma_{\xi,n})(x)\,\mu_{L_n}(dx), \quad \gamma_{\xi,n}(y) := \prod_{l=1}^{m(n)} \gamma_\xi(y_l),$$

F_n is a Fourier transformation by $(y_1,\dots,y_n)$, $y = (y_j : j \in \mathbf{N})$, $y_j \in \mathbf{K}$, $\gamma_\xi(y_l)$ are the same as in Lemma 2.8 *for* $\mathbf{K}^1$*; here $m(n) = dim_{\mathbf{K}} L_n < \aleph_0$, $cl(\bigcup_n L_n) = X = c_0(\omega_0, K)$.*

Proof. If a sequence of weak distributions is generated by a measure μ, then in view of Conditions 2.6(3-6), Lemmas 2.3, 2.5, 2.8, Propositions 2.10 and 2.16, Corollary 2.13, the Lebesgue convergence theorem and the Fubini theorem, also from the proof of Theorem 2.27 and the Radon property of μ it follows that there exists $r > 0$ such that

$$\int_X G_\xi(x)\mu_*(dx) = \int_X G_\xi(x)\mu(dx) = \lim_{n\to\infty} \int_{L_n} \gamma_{\xi,n}(y)\hat{\mu}_{L_n}(y)m_{L_n}(dy),$$

since $\lim_{j\to\infty} x_j = 0$ for each $x = (x_j : j) \in X$. In addition, $\lim_{|\xi|\to\infty} S_\xi(\{\mu_{L_n}\}) = \int_X \mu(dx) = 1$. Indeed, for each $c > 0$ and $d > 0$ there exists a compact $V_c \subset X$ with $\|\mu|_{(X\setminus V_c)}\| < c$ and there exists n_0 with $V_c \subset L_n^d$ for each $n > n_0$. Therefore, choosing suitable sequences of $c(n)$, $d(n)$, $V_{c(n)}$ and L_{j_n} we get that $[\int_{L_n} \gamma_{\xi,n}(y)\hat{\mu}_{L_n}(y)m_{L_n}(dy) : n \in \mathbf{N}]$ is a Cauchy sequence, where m_{L_n} is the real Haar measure on L_n, the latter is considered as $\mathbf{Q_p^{m(n)b}}$, $b = dim_{\mathbf{Q_p}}\mathbf{K}$, $m(B(L_n,0,1)) = 1$. Here we use $G_\xi(x)$ for a formal expression of the limit S_ξ as the integral. Then $G_\xi(x)$ $(mod\ \mu)$ is defined evidently as a function for μ or $\{\mu_{L_n} : n\}$ with a compact support, also for μ with a support in a finite-dimensional subspace L over $\mathbf{K}$ in X. By the definition $supp\ (\mu_{L_n} : n)$ is compact, if there is a compact $V \subset X$ with $supp(\mu_{L_n}) \subset P_{L_n}V$ for each n. That is, Condition (i) is necessary.

On the other hand, if (i) is satisfied, then for each $c > 0$ there exists $r > 0$ such that $|\int_X G_\xi(x)\mu_*(dx) - 1| < c/2$, when $|\xi| > r$, consequently, there exists n_0 such that for each $n > n_0$ the following inequality is satisfied:

$$\left|1 - \int_X F_n(\gamma_{\xi,n})(x)\mu_*(dx)\right| \leq |\|\mu|_{(L_n\cap B(X,0,R))}\| - 1|$$

$$+ \sup_{|x|>R} |F_n(\gamma_{\xi,n})(x)|\|\mu_{L_n}|_{(L_n\setminus B(X,0,R))}\|.$$

Therefore, from $\lim_{R\to\infty}\sup_{|x|>R} |F_n(\gamma_{\xi,n})(x)| = 0$ and from Lemma 2.3 the statement of Lemma 2.30 follows.

2.31. Notes and definitions. Suppose X is a locally convex space over a locally compact field $\mathbf{K}$ with non-trivial non-Archimedean normalization and X^* is a topologically dual space. The minimum σ-algebra with respect to which the following family $\{v^* : v^* \in X^*\}$ is measurable is called a σ-algebra of cylindrical sets. Then X is called a *RS*-space if on X^* there exists a topology τ such that the continuity of each positive definite function $f : X^* \to \mathbf{C}$ is necessary and sufficient for f to be a characteristic functional of a non-negative measure. Such topology is called the *R*-Sazonov type topology. The class of *RS*-spaces contains all separable locally convex spaces over $\mathbf{K}$. For example, $l^\infty(\alpha,\mathbf{K}) = c_0(\alpha,\mathbf{K})^*$, where α is an ordinal [Roo78]. In particular we also write $c_0(\mathbf{K}) := c_0(\omega_0,\mathbf{K})$ and $l^\infty(\mathbf{K}) := l^\infty(\omega_0,\mathbf{K})$, where ω_0 is the first countable ordinal.

Let $n_{\mathbf{K}}(l^\infty, c_0)$ denotes the weakest topology on l^∞ for which all functionals $p_x(y) := \sup_n |x_n y_n|$ are continuous, where $x = \sum_n x_n e_n \in c_0$ and $y = \sum_n y_n e_n^* \in l^\infty$, e_n is the standard base in c_0. Such topology $n_{\mathbf{K}}(l^\infty, c_0)$ is called the normal topology. The induced topology on c_0 is denoted by $n_{\mathbf{K}}(c_0, c_0)$.

2.32. Theorem. *Let $f : l^\infty(\mathbf{K}) \to \mathbf{C}$ be a functional such that*

(i) f is positive definite,

(ii) f is continuous in the normal topology $n_{\mathbf{K}}(l^\infty, c_0)$, then f is the characteristic functional of a probability measure on $c_0(\mathbf{K})$.

Proof. The case of the topological vector space X over $\mathbf{K}$ with $char(\mathbf{K}) > 0$ and a real-valued measure μ can be proved analogously to the proofs in Chapter II for $\mathbf{K_s}$-valued measures as well as for $char(\mathbf{K}) = 0$ due to §2.6 and §§2.25-2.30 (see also [Mad85]).

2.33. Theorem. *Let μ be a probability measure on $c_0(\mathbf{K})$, then $\hat{\mu}$ is continuous in the normal topology $n_{\mathbf{K}}(l^\infty, c_0)$ on l^∞.*

Proof. In view of Lemma 2.3 for each $\varepsilon > 0$ there exists $S(\varepsilon) \in c_0$ such that $\|\mu|_{L(0,S(\varepsilon))}\| \geq 1 - \varepsilon$, where $L(y,z) := \{x \in c_0 : |x_n - y_n| \leq |z_n|$, for each $n \in \mathbf{N}\}$. Therefore,

$$|1 - \hat{\mu}(x)| \leq \varepsilon + \|2\pi\eta(\xi x)\|_{C^0(L(0,S(\varepsilon)))}\|\mu|_{L(0,S(\varepsilon))}\|,$$

hence there exists a constant $C > 0$ such that $|1 - \hat{\mu}| \leq \varepsilon + Cp_{S(\varepsilon)}(x)$.

2.34. Corollary. *The normal topology $n_{\mathbf{K}}(l^\infty, c_0)$ is the R-Sazonov type topology on $l^\infty(\mathbf{K})$.*

2.35. Theorem. Non-Archimedean analog of the Minlos-Sazonov theorem. *For a separable Banach space X over $\mathbf{K}$ the following two conditions are equivalent:*

$$(I)\ \theta : X \to \mathbf{T} \textit{ satisfies conditions } 2.6(3,4,5) \textit{ and}$$

for each $c > 0$ there exists a compact operator $S_c : X \to X$ such that $|Re(\theta(y) - \theta(x))| < c$ for $|\tilde{z}(S_c z)| < 1$;

$$(II)\ \theta \textit{ is a characteristic functional of a probability Radon measure } \mu$$

on E, where $\tilde{z}$ is an element $z \in X \hookrightarrow X^*$ considered as an element of X^* under the natural embedding associated with the standard base of $c_0(\omega_0, \mathbf{K})$, $z = x - y$, x and y are arbitrary elements of X.

Proof. $(II \to I)$. For a positive definite function θ generated by a probability measure μ in view of the inequality $|\theta(y) - \theta(x)|^2 \leq 2\theta(0)(\theta(0) - Re(\theta(y - x))$ (see also Propositions

IV.1.1(c)[VTC85]) and using the normalization of a measure μ by 1 we consider the case $y=0$. For each $r>0$ we have:

$$|Re(\theta(0)-\theta(x))| = \int_X (1-[\chi_e(x(u))+\chi_e(-x(u))]/2)\mu(du)$$

$$\leq \int_{B(X,0,r)} 2[\chi_e^{1/2}(x(u))-\chi_e^{1/2}(-x(u))/(2i)]^2\mu(du)+2\int_{X\setminus B(X,0,r)}\mu(du)$$

$$\leq 2\pi^2\int_{B(X,0,r)}\eta(x(u))^2\mu(dx)+2\mu([x: \quad \|x\|>r]).$$

In view of the Radon property of the space X and Lemma 2.5 for each $b>0$ and $\delta>0$ there are a finite-dimensional over $\mathbf{K}$ subspace L in X and a compact subset $W\subset X$ such that $W\subset L^\delta$, $\|\mu|_{(X\setminus W)}\|<b$, hence $\|\mu|_{(X\setminus L^\delta)}\|<b$.

We consider the following expression:

$$J(j,l) := 2\pi^2\int_{B(X,0,r)}\eta(e_j(u))\eta(e_l(u))\mu(du),$$

where (e_j) is the orthonormal basis in X which contains the orthonormal basis of $L=\mathbf{K^n}$, $n=dim_{\mathbf{K}}L$. Then we choose sequences $b_j=p^{-j}$ and $0<\delta_j<b_j$, subspaces L_j and $r=r_j$ such that $b_jr_j<1$, $W_j\subset B(X,0,r_j)$, $0<r_j<r_{j+1}<\infty$ for each $j\in\mathbf{N}$ and the orthonormal basis (e_j) corresponding to the sequence $L_j\subset L_{j+1}\subset\cdots\subset X$. We get, due to finiteness of $n_j := dim_{\mathbf{K}}L_j$, that $\lim_{j+l\to\infty}J(j,l)=0$, since $\|\mu|_{\{x:\|x\|>r_j\}}\|<b_j$, $\eta(x(u))=0$ for $x\in X\ominus L_j$ with $\|x\|<b_j$, $u\in B(X,0,r_j)$. Then we define $g_{j,l} := \min\{d: d\in\Gamma_{\mathbf{K}}$ and $d\geq|J(j,l)|\}$, evidently, $g_{j,l}\leq p|J(j,l)|$ and there are $\xi_{j,l}\in\mathbf{K}$ with $|\xi_{j,l}|_{\mathbf{K}}=g_{j,l}$. Consequently, the family $(\xi_{j,l})$ determines a compact operator $S:X\to X$ with $\tilde{e}_j(Se_l)=\xi_{j,l}t$ due to Theorem 1.2[Sch89], where $t=const\in\mathbf{K}$, $t\neq 0$. Therefore, $|Re(\theta(0)-\theta(z))|<c/2+|\tilde{z}(Sz)|<c$ $|\theta(0)-\theta(z)|<c/2+|\tilde{z}(Sz)|<c$. We choose r such that $\|\mu|_{(X\setminus B(X,0,r))}\|<c/2$ with S corresponding to $(r_j:j)$, where $r_1=r$, $L_1=L$, then we take $t\in\mathbf{K}$ with $|t|c=2$.

$(I\to II)$. Without restriction of generality we may take $\theta(0)=1$ after renormalization of non-trivial θ. In view of Theorem 2.32 as in §2.6 we construct using $\theta(z)$ a consistent family of finite-dimensional distributions $\{\mu_{L_n}\}$. Let m_{L_n} be a real Haar measure on L_n which is considered as $\mathbf{Q_p^a}$ with $a=dim_{\mathbf{K}}L_n dim_{\mathbf{Q_p}}\mathbf{K}$, $m(B(L_n,0,1))=1$. In view of Proposition 2.7 and Lemmas 2.8, 2.30:

$$\int_{L_n}G_\xi(x)\mu_{L_n}(dx)=\int_{L_n}\gamma_{\xi,n}(z)\theta(z)m_{L_n}(dz),$$

consequently,

$$1-\int_{L_n}F_n(\gamma_{\xi,n})(x)\mu_{L_n}(dx)=\int\gamma_{\xi,n}(z)(1-\theta(z))m_{L_n}(dz)=:I_n(\xi).$$

There exists an orthonormal basis in X in which S_c can be reduced to the following form $S_c=SC\hat{S}_cE$ (see Appendix), where $\hat{S}_c=diag(s_j:j\in\mathbf{N})$ in the orthonormal basis $(f_j:j)$ in X and S transposes a finite number of vectors in the orthonormal basis. That is, $|\tilde{z}(\hat{S}_cz)|=\max_j|s_j|\times|z_j|^2$. In the orthonormal basis $(e_j:j)$ adopted to $(L_n:n)$ we have $|\tilde{z}(S_cz)|=\max_{j,l\in\mathbf{N}}(|s_{j,l}|\times|z_j|\times|z_l|)$, $\|S_c\|=\max_{j,l}|s_{j,l}|$, where $S_c=(s_{j,l}:j,l\in\mathbf{N})$ in the orthonormal basis (e_j), $r = const > 0$. In addition, $p^{-1}|x|_K \le |x|_p \le p|x|_{\mathbf{K}}$ for each $x \in \mathbf{K}$. If S_c is a compact operator such that $|Re(\theta(y) - \theta(x))| < c$ for $|\tilde{z}(S_c z)| < 1$, $z = x - y$, then $|Re(1 - \theta(x))| < c + 2|\tilde{x}(S_c x)|$ and

$$I_n(\xi) \le \int \gamma_{\xi,n}(z)[c + 2|\tilde{z}(S_c z)|_K] m_{L_n}(dz) \le c + b\|S_c\|/|\xi|^2,$$

$b = const$ is independent from n, ξ and S_c,

$$b := p \times \sup_{|\xi|>r} |\xi|^2 \int_{L_n} \gamma_{\xi,n}(z)|z|_p^2 m_{L_n}(dz) < \infty.$$

Due to the formula of changing variables in integrals (A.7[Sch84]) the following equality is valid: $J_n(\xi) = I_n(\xi)J_n(1)/[I_n(1)|\xi|^2]$ for $|\xi| \ne 0$, where

$$J_n(\xi) = \int_{L_n} \gamma_{\xi,n}(z)|z|_p^2 m_{L_n}(dz).$$

Therefore,

$$1 - \int_X G_\xi(x)\mu_*(dx) \le c + b\|S_c\|/|\xi|^2.$$

Then taking the limit with $|\xi| \to \infty$ and then with $c \to +0$ with the help of Lemma 2.30 we get the statement $(I \to II)$.

2.36. Definition. Let on a completely regular space X with the small inductive dimension $ind(X) = 0$ two non-zero real-valued measures μ and ν be given. Then ν is called absolutely continuous relative to μ if $\nu(A) = 0$ for each $A \in Bf(X)$ with $\mu(A) = 0$ and it is denoted $\nu \ll \mu$. Measures ν and μ are singular to each other if there is $F \in Bf(X)$ with $|\mu|(X \setminus F) = 0$ and $|\nu|(F) = 0$ and it is denoted $\nu \perp \mu$. If $\nu \ll \mu$ and $\mu \ll \nu$ then they are called equivalent, $\nu \sim \mu$.

2.37. For a Banach space X over a non-Archimedean infinite locally compact field $\mathbf{K}$ and σ-algebra $\mathsf{B} \supset Bf(X)$ with a real probability measure μ and a σ-subalgebra $\mathsf{B}_0 \subset \mathsf{B}$ a function $\bar{\mu}(A|x)$ satisfying three conditions:

(a) $\bar{\mu}(A|x)$ is B_0-measurable by x for each $A \in \mathsf{B}$;

(b) $\mu(A \cap A_0) = \int_{A_0} \bar{\mu}(A|x)\mu(dx) =: \mu_A(A_0)$ for each $A_0 \in \mathsf{B}_0$;

(c) $\bar{\mu}(A|x)$ is a measure by $A \in \mathsf{B}$ for almost all x relative to a measure μ, then it is called a conditional measure corresponding to μ relative to a σ-algebra B_0.

Then we define the conditional measure $\mu(A|x)$ by Formula (b) and we then redefine it on the set of measure zero relative to μ for each A. Let $\theta(z|x) = \theta_1(z|x) + i\theta_2(z|x)$ with

$$\int_{A_0} [[\chi_e(\tilde{z}(x)) + \chi_e(-\tilde{z}(x))]/2]\mu(dx) = \int_{A_0} \theta_1(z|x)\mu(dx);$$

$$\int_{A_0} [\chi_e(\tilde{z}(x)) - \chi_e(-\tilde{z}(x))]/(2i)]\mu(dx) = \int_{A_0} \theta_2(z|x)\mu(dx),$$

where θ_j are measurable functions, $z \in X$, $\tilde{z} \in X'$ corresponds to z under the natural embedding of X into $X' = X^*$, where X^* is the topologically dual space of X, $x \in X$, $A_0 \in \mathsf{B}_0$. Then we choose sets $A_{n,k}$ with of diameters $diam(A_{n,k}) < p^{-n}$, using §§2.7, 2.8, 2.30 and the proof of §2.27, using the function $G_\xi(x)$, defining sets B_n, G, G_1, we get for each $c>0$ that there exists $r>0$ such that

$$Re(1-\theta(z|x)) \le \int_X G_\xi(y)(1-[[\chi_e(\tilde{z}(y))+\chi_e(-\tilde{z}(y))]/2]\mu(dy|x)+2\int_X(1-G_\xi(y))\mu(dy|x)$$

$$\le \int (2\pi\eta^2(\tilde{z}(y))G_\xi(y)\mu(dy|x)/2+c/2$$

for each $|\xi|>r$ and $x\in G_1\cap G$, since from $\|z\|\times\|y\|\le 1$ it follows that $(1-\chi_e(\tilde{z}(y))=0$ and $|1-\exp(it)|\le|t|$ for $t\in\mathbf{R}$, $i=(-1)^{1/2}$.

For $C(z_1,z_2)(x) := \int_X \eta(\tilde{z}_1(y))\eta(\tilde{z}_2(y))G_\xi(y)\mu(dy|x)$ we have $C(0,0)=0$ and $C(z,z)(x) = \int_{X\setminus B(X,0,r)}\eta^2(\tilde{z}(y))G_\xi(y)\mu(dy|x)$ for $\|z\|=1/r>0$. Since $\theta(z|x) = \int_X \chi_e(\tilde{z}(y))\mu(dy|x)$, considering projectors $P_{L_n}: X\to L_n$ onto finite-dimensional subspaces over $\mathbf{K}$, $L\subset X$, we get $\theta_L(z|x)$ corresponding to $\mu_L(dx)$ and $\mu_L(dy|x)$. At the same time θ_L satisfies Conditions 2.6(3,4,5), consequently, $\mu_L(A|x)$ are σ-additive by A for fixed x, since $\theta(z|x)$ are continuous by z for $x\in G\cap G_1$.

Remind the Prohorov's theorem. Let $\mathcal{T}=(T_k,p_{k,j})$ be a projective system of topological spaces, where $k,j\in J$, let also T be a topological space and $(p_j: j\in J)$ be a cogerent $p_k=p_{k,j}\circ p_j$ defining family of continuous mappings $p_j: T\to T_j$, $p_{k,j}: T_j\to T_k$ for each $k\le j\in J$. Suppose that $(\mu_j: j\in J)$ is a projective system of measures on $\mathcal{T}$, $\mu_k=p_{k,j}(\mu_j)$ is a bounded measure on T_k for each $k\le j\in J$. Then a bounded measure μ on T with $p_j(\mu)=\mu_j$ for each $j\in J$ exists if and only if the following condition is satisfied:

(P) for each $c>0$ there exists a compact subset $K\subset T$ so that $\mu_j^*(T_j\setminus p_j(K))\le c$ for each $j\in J$. If this condition (P) is satisfied, then μ is unique and $\mu^*(K)=\inf_j\mu_j^*(p_j(K))$ for each compact subset $K\subset T$ (see also §IX.4.2.1 in [Bou63-69]).

From the Prohorov's theorem, Lemma 2.5, $x\in G\cap G_1$ and

$$C(e_j,e_l)(x)=\int_{L_{j,l}\setminus B(L_{j,l},0,1)}\eta(y_l)\eta(y_j)F_2(\gamma_{\xi,2})(y_j,y_l)\mu_{L_{j,l}}(dy_jdy_l|x)$$

it follows that

$$\lim_{j+l\to\infty}\int_{L_{j,l}}\psi(x)C(e_j,e_l)(x)\mu_{L_{j,l}}(dx)$$

$$=\lim_{j+l\to\infty}\int_{L_{j,l}}\psi(x)\int_{L_{j,l}}\eta(y_j)\eta(y_l)\chi_e(\tilde{z}(y))\mu_{L_{j,l}}(dy|x)\mu_{L_{j,l}}(dx)=0$$

for each bounded B_0-measurable function $\psi(x)$, whence $\lim_{n\to\infty}\mu([x: |C(e_j,e_l)(x)|>c$ for some $j+l>n])=0$ for each $c>0$, where $L_{j,l}=\mathbf{K}e_j\oplus\mathbf{K}e_l$, $y_j\in\mathbf{K}$.

Let $G_2 := [x:$ for each $c>0$ there exists $n>0$ with $|C(e_j,e_l)(x)|<c$ for each $j+l>n]\subset X\setminus(\bigcup_{c>0}\bigcap_{n=1}^\infty\bigcup_{j+l>n}[x: |C(e_j,e_l)(x)|>c])$, then $\mu(G_2)=1$. Therefore, for each $x\in G\cap G_1\cap G_2$ there exists a compact operator $D: X\to X$ such that $|C(z_1,z_2)(x)|\le|\tilde{z}_2(Dz_1)|$ for each z_1, $z_2\in X$.

In view of Theorem 2.35 and the equality

$$\int_{A_0}[\int_X\chi_e(z(y))\bar{\mu}(dy|x)]\mu(dx)=\int_{A_0}\chi_e(z(y))\bar{\mu}_{A_0}(dy)$$

we get that

$$\int_X\chi_e(z(y))\bar{\mu}_{A_0}(dy)=\int_X\theta(z,x)\mu(dx)=\int_{A_0}\chi_e(z(y))\mu(dy),$$

where $\bar{\mu}_{A_0}(B) = \mu(A_0|B)$ is the measure on $Bf(X)$, so that $\bar{\mu}_{A_0}(B) = \mu(A_0 \cap B)$. That is

$$\mu(A_0 \cap B) = \int_{A_0} \bar{\mu}(B|x)\mu(dx), \tag{1}$$

hence $\bar{\mu}(B|x)$ is the conditional measure of μ relative to the σ-algebra B_0. Thus the existence of the conditional measure is demonstrated.

There is the following important particular case, when the σ-algebra B_0 is generated by some finite or infinite family of functions $\{\phi_j : j \in J\}$, that is is the minimal σ-algebra relative to which all these functions are measurable.

The relation (1) is equivalent with the statement:

for each bounded B_0-measurable function $\psi(x)$ and $Bf(X)$-measurable function $\psi(x)$, for which $\int_X |\phi(x)|\mu(dx) < \infty$ the equality

$$\int_X \psi(x)\phi(x)\mu(dx) = \int_X \psi(x) \int_X \phi(y)\bar{\mu}(dy|x)\mu(dx) \tag{2}$$

is satisfied. Relation (2) implies that

$$\int_X g(y)\phi(y)\bar{\mu}(dy|x) = g(x) \int_X \phi(y)\bar{\mu}(dy|x) \quad (mod\ \mu) \tag{3}$$

for each bounded B_0-measurable function $g(x)$ and a function $\phi(x)$, for which $\int_X |\phi(x)|\mu(dx) < \infty$.

2.38. Definition. Martingales. Let $(X, \mathcal{B}, \mu)$ be a measure space, where μ is a non-negative measure on a measurable space $(X, \mathcal{B})$, $\mathcal{B}$ is a σ-algebra on a set X. A sequence of measurable real-valued functions $\{\phi_n(x) : n \in \mathbf{N}\}$ on $(X, \mathcal{B}, \mu)$ is called a martingale, if for each n:

$$\int_X |\phi_n(x)|\mu(dx) < \infty, \tag{1}$$

while for each $\mathcal{U}_n$ measurable bounded non-negative function $\psi(x) \geq 0$, the relation

$$\int_X \phi_{n+1}(x)\psi(x)\mu(dx) = \int_X \phi_n(x)\psi(x)\mu(dx) \tag{2}$$

is satisfied, where $\mathcal{U}_n$ denotes the minimal σ-algebra relative to which functions $\phi_1, \ldots, \phi_n$ are measurable.

If instead of the equality there is the inequality $\geq$ or $\leq$, then such sequence is called the sub-martingale or super-martingale respectively.

2.39. Lemma. *For each sub-martingale (super-martingale) on a Banach space X over* $\mathbf{K}$ *there exists a sequence of functions $g_n(x)$ satisfying the following conditions $(a-c)$:*

(a) $g_n(x)$ are $\mathcal{U}_n$-measurable;

(b) the sequence $\phi_n - g_n$ is the martingale;

(b) $g_n(x)$ increases with n (decreases with n correspondingly) μ-almost everywhere.

Proof. Put

$$(i) \quad g_{n+1}(x) - g_n(x) := \int_X \phi_{n+1}(y)\mu(dy, \mathcal{U}_n|x) - \phi_n(x),$$

where $\mu(A, \mathcal{U}_n|x)$ is the conditional measure of μ with respect to the σ-algebra $\mathcal{U}_n$. The existence fo such conditional measure is proved in §2.37. Evidently, $g_{n+1}(x) - g_n(x)$ is the $\mathcal{U}_n$-measurable function. Multiplying this function on any bounded non-negative $\mathcal{U}_n$-measurable function $\psi(x)$ due to Equality 2.37(2) we deduce that

$$(ii) \quad \int_X [g_{n+1}(x) - g_n(x)]\psi(x)\mu(dx) = \int_X \int_X \phi_{n+1}(y)\mu(dy, \mathcal{U}_n|x)\psi(x)\mu(dx)$$

$$- \int_X \phi_n(x)\psi(x)\mu(dx) = \int_X \phi_{n+1}(x)\psi(x)\mu(dx) - \int_X \phi_n(x)\psi(x)\mu(dx).$$

Form this equality it follows, that

$$(iii) \quad \int_X [g_{n+1}(x) + \phi_{n+1}(x)]\psi(x)\mu(dx) = \int_X [g_n(x) + \phi_n(x)]\psi(x)\mu(dx),$$

since $g_{n+1}(x) = \sum_{k=1}^n [g_{k+1}(x) - g_k(x)]$, with $g_1(x) := 0$. Let in Formula (ii) be $\psi(x) = 1$, when $g_{n+1}(x) - g_n(x) < 0$ and $\psi(x) = 0$ in other cases. Then from the definition of the sub-martingale it follows that

$$- \int_X \psi(x)\mu(dx) = \int_X [g_{n+1}(x) - g_n(x)]\psi(x)\mu(dx) \geq 0,$$

consequently, $g_{n+1}(x) \geq g_n(x)$ μ-almost everywhere.

2.40. Remark. From the proof of the preceding lemma we get that

$$\int_X \phi_{n+1}(y)\mu(dy, \mathcal{U}_n|x) \geq \phi_n(x) \tag{1}$$

for each n. In the martingale case we have here the equality instead of the inequality. If $\int_X |\phi_n(x)|\mu(dx) < \infty$ for each n, then $\{\phi_n : n\}$ is the sub-martingale. If $g_n(x)$ are the same as in §2.39, then

$$\int_X g_n(x)\mu(dx) = \int_X [g_n(x) - \phi_n(x)]\mu(dx) + \int_X \phi_n(x)\mu(dx).$$

Taking in 2.38(2) $\psi(x) = 1$ we get that $\int_X f_n(x)\mu(dx) = \int_X f_1(x)\mu(dx)$ for the martingale $\{f_n(x) = g_n(x) - \phi_n(x) : n\}$. Therefore, $\int_X g_n(x)\mu(dx) = \int_X [g_1(x) - \phi_1(x)]\mu(dx) + \int_X \phi_n(x)\mu(dx) = \int_X \phi_n(x)\mu(dx) - \int_X \phi_1(x)\mu(dx)$. If for a sub-martingale the condition

$$\sup_n \int_X |\phi_n(x)|\mu(dx) < \infty \tag{2}$$

is satisfied, then

$$\sup_n \int_X g_n(x)\mu(dx) < \infty, \tag{3}$$

hence a non-decreasing sequence $g_n(x)$ has μ-almost everywhere a finite limit. Moreover, this linmit is μ-integrable. If $\{\phi_n : n\}$ is a super-martingale, then $\{-\phi_n : n\}$ is a sub-martingale and for it an integrable limit of the sequence $-g_n(x)$ exists. Thus for the proof of an existence of the limit

$$\lim_{n\to\infty} \phi_n(x) \tag{4}$$

for martingales, sub-martingales and super-martingales the consideration of martingales only is sufficient.

2.41. Lemma. *If* $\{\phi_n : n\}$ *is a martingale, then for all n and* $b > 0$ *the inequality*

$$\mu(\{x : \sup_k \phi_k(x) \geq b\}) \leq \sup_n \int_X \phi_n^+(x)\mu(dx)/b \qquad (1)$$

is satisfied, where $\phi_n^+(x) = \phi_n(x)$ *for* $\phi_n(x) \geq 0$*, while* $\phi_n^+(x) = 0$ *for* $\phi_n(x) \leq 0$*. If* $\sup_n \int_X \phi_n(x)\mu(dx) < \infty$*, then*

$$\mu(\{x : \sup_k \phi_k(x) \geq b\}) \leq \sup \int_X \phi_n^+(x)\mu(dx)/b. \qquad (2)$$

Proof. Let $\chi_k(x) = 1$, when $\phi_1(x) < b,\dots,\phi_{k-1}(x) < b$, $\phi_k(x) \geq 0$, $\phi_k(x) = 0$ in other cases, hence $\phi_k(x)$ is the $\mathcal{A}_k$-measurable function. Therefore,

$$\int_X \phi_n(x)\chi_k(x)\mu(dx) = \int_X \phi_{n-1}(x)\chi_k(x)\mu(dx)$$

$$= \cdots = \int_X \phi_k(x)\chi_k(x)\mu(dx) \geq b\int_X \chi_k(x)\mu(dx)$$

for each $k \leq n$, consequently,

$$\int_X \phi_n(x)\sum_{k=1}^n \chi_k(x)\mu(dx) \geq b\int_X \sum_{k=1}^n \chi_k(x)\mu(dx). \qquad (3)$$

On the other hand, the function $\sum_{k=1}^n \chi_k(x)$ is characteristic of the set $\{x \in X : \sup_{k\leq n}\phi_k(x) \geq b\} = B_n$. From Equation 2.38(2) we infer that

$$\mu(B_n) \leq \int_{B_n} \phi_n(x)\mu(dx)/b \leq \int_X \phi_n^+(x)\mu(dx)/b.$$

This demonstrates Inequality (1). Taking the limit by n tending to the infinity we get (2) as well.

2.42. Theorem. *If* $\{\phi_n(x) : n\}$ *is a martingale for which Condition* 2.41(3) *is satisfied, then* μ*-almost everywhere the limit* 2.40(4) *exists.*

Proof. We shall say that a sequence $\alpha_1,\dots,\alpha_2,\dots$ crosses infinite times the segment $[\beta_1,\beta_2]$ with $\beta_1 < \beta_2$, if it is possible to choose $k_1 < k_2 < \cdots$ so that $\alpha_{k_1} \geq \beta_2$, $\alpha_{k_2} \leq \beta_1,\dots$, $\alpha_{k_{2n-1}} \geq \beta_2$, $\alpha_{k_{2n}} \leq \beta_1,\dots$.

Consider the set B_{β_1,β_2} consisting of all $x \in X$ for which the sequence $\{\phi_n(x) : n\}$ crosses infinite number of times the segment $[\beta_1,\beta_2]$. Denote by B_- the set of all $x \in X$ for which $\inf_n \phi_n(x) = -\infty$, while B_+ is the set of all $x \in X$ with $\sup_n \phi_n(x) = \infty$. Therefore, the set $B_+ \cup B_- \cup [\bigcup_{\beta_1<\beta_2\in\mathbf{Q}} B_{\beta_1,\beta_2}]$ consists of all $x \in X$ for which the limit 2.40(4) does not exist. In accordance with 2.41(2) we get $\mu(B_+) = \lim_{b\to\infty}\mu(\{x \in X :\ \sup_k \phi_k(x) \geq b\}) = 0$. Considering the sequence $-\phi_n(x)$ we find that $\mu(B_-) = 0$. Therefore, it remains to show that $\mu(B_{\beta_1,\beta_2}) = 0$ for each pair of rational numbers $\beta_1 < \beta_2$.

Now we define the following sequence of functions $k_n(x)$ as: $k_0(x) = 0$, $k_1(x) = j$ if $\phi_l(x) < \beta_2$ for each $l < j$ while $\phi_j(x) \geq \beta_2$; $k_1(x) = \infty$ if $\phi_l(x) < \beta_2$ for all $l > 0$. Then $k_2(x) = j$ if $k_1(x) < \infty$ and $\phi_p(x) > \beta_1$ and $\phi_j(x) \leq \beta_1$ for $k_1(x) \leq p < j$. In others cases we put $k_2(x) = \infty$. If $k_{2n}(x)$ is defined, then $k_{2n+1}(x) = j$ if $k_{2n}(x) < \infty$ and $\phi_p(x) < \beta_2$ while $\phi_j(x) \geq \beta_2$ for $k_{2n}(x) \leq p < j$. If either $k_{2n}(x) = \infty$ or $\phi_p(x) > \beta_2$ for $p > k_{2n}(x)$, then put

$k_{2n+1} = \infty$. If $k_{2n+1}(x)$ is defined, then $k_{2n+2}(x) = j$. If $k_{2n+1}(x) < \infty$ and $\phi_p(x) > \beta_1$ and $\phi_j(x) \le \beta_1$ for $k_{2n+1}(x) \le p < j$, then $k_{2n+2}(x) = j$. If either $k_{2n+1}(x) = \infty$ or $\phi_p(x) > \beta_1$ for $p > k_{2n+1}(x)$, then $k_{2n+2}(x) = \infty$.

We put further $\chi_n(x) = 1$ when $k_{2p}(x) \le n < k_{2p+1}(x)$ for some p, while $\chi_n(x) = -1$ when $k_{2p-1}(x) \le n < k_{2p}(x)$ for some p. Therefore, functions $\chi_n(x)$ are completely defined by $\phi_1(x), \dots, \phi_n(x)$ so that χ_n is the Borel function of $\phi_1, \dots, \phi_n$. Thus χ_n is $\mathcal{U}_n$-measurable.

Construct the functions

$$g_n(x) := \phi_1(x) + \sum_{k=1}^{n-1} [\phi_{k+1}(x) - \phi_k(x)]\chi_k(x).$$

We shall show that $\{g_n(x) : n\}$ is the martingale. Denote by $\mathcal{P}_n$ the σ-algebra generated by functions $g_1(x), \dots, g_n(x)$. Since $g_k(x)$ is $\mathcal{U}_k$-measurable, then $\mathcal{P}_n \subset \mathcal{U}_n$ for each n. If a function $\psi(x)$ is bounded and $\mathcal{P}_n$-measurable ,then it is $\mathcal{U}_n$-measurable. This implies that

$$\int_X g_{n+1}(x)\psi(x)\mu(dx)$$

$$= \int_X [g_n(x) + (\phi_{n+1}(x) - \phi_n(x))\chi_n(x)]\psi(x)\mu(dx) = \int_X g_n(x)\psi(x)\mu(dx),$$

since $\int_X [\phi_{n+1}(x) - \phi_n(x)]\chi_n(x)\psi(x)\mu(dx) = 0$ due to $\mathcal{U}_n$-measurability and boundedness of $\chi_n(x)\psi(x)$, consequently, $g_n(x)$ is the martingale.

From $g_1(x) = \phi_1(x)$ it follows that

$$\int_X g_n(x)\mu(dx) = \int_X g_1(x)\mu(dx) = \int_X \phi_1(x)\mu(dx).$$

For each $n \le k_1(x)$ we have $g_n(x) = \phi_n(x)$, while for each $k_1(x) < n \le k_2(x)$ there is the equality $g_n(x) - g_{k_1(x)}(x) = \phi_{k_1(x)}(x) - \phi_n(x)$ or $g_n(x) = 2\phi_{k_1(x)}(x) - \phi_n(x) \ge 2\beta_2 - \phi_n(x)$. This means that $g_{k_2(x)}(x) \ge 2\beta_2 - \beta_1 > \beta_1 \ge \phi_{k_2(x)}(x)$ and $g_n(x) \ge \phi_n(x)$ for each $k_2(x) < n \le k_3(x)$. When the inequalities $k_3(x) < n \le k_4(x)$ are satisfied, then $g_n(x) - g_{k_3(x)}(x) = \phi_{k_3(x)}(x) - \phi_n(x)$ also $g_n(x) \ge 2\phi_{k_3(x)}(x) - \phi_n(x) \ge 2\beta_2 - \phi_n(x)$. So we deduce that

$$g_n(x) \ge \min_n [\phi_n(x), 2\beta_2 - \phi_n(x)]. \tag{1}$$

Consider now the functions $\phi_n = \phi_n^+ - \phi_n{}^-$, $g_n = g_n^+ - g_n^-$, where $g_n^+(x)g_n^-(x) = 0$, $\phi_n^+(x)\phi_n^-(x) = 0$, $\phi_n^+(x) \ge 0$, $\phi_n^-(x) \ge 0$, $g_n^+(x) \ge 0$, $g_n^-(x) \ge 0$. From Inequality (1) it follows that

$$g_n^-(x) = \max[g_n^-(x), g_n^+(x) - 2\beta_2] \le |\phi_n(x)| + 2|\beta_2|.$$

Therefore,

$$\int_X g_n^+(x)\mu(dx) = \int_X g_n(x)\mu(dx) + \int_X g_n^-(x)\mu(dx)$$

$$\le \int_X [\phi_n(x) + |\phi_n(x)| + 2|\beta_2|]\mu(dx) \le 2\int_X g_n^+(x)\mu(dx) + 2|\beta_2|,$$

consequently,

$$\mu(\{x \in X : \sup_n g_n(x) = \infty\}) = \lim_{b\to\infty} \mu(\{x \in X : \sup_n g_n(x) \ge b\})$$

$$\leq \lim_{b\to\infty} \sup_n \int_X g_n^+(x)\mu(dx)/b = 0.$$

We have that $k_n(x) < \infty$ for each n when $x \in B_{\beta_1,\beta_2}$, hence $B_{\beta_1,\beta_2} \subset \{x \in X : \ \sup_n g_n(x) = \infty\}$. If $k_n(x) < \infty$, then $g_{k_n(x)}(x) - g_{k_{n-1}(x)}(x) \geq \beta_2 - \beta_1$ and $g_{k_1(x)}(x) \geq \beta_2$, consequently, $g_{k_n(x)}(x) \geq \beta_2 + (n-1)[\beta_2 - \beta_1]$ and inevitably $\sup_n g_n(x) = \infty$ for $x \in B_{\beta_1,\beta_2}$. Thus $\mu(B_{\beta_1,\beta_2}) = 0$.

2.43. Corollary. *Each non-negative martingale $\{\phi_n(x) : n\}$ has a limit almost everywhere by the measure μ. If $\phi(x) = \lim_{n\to\infty} \phi_n(x) \pmod{\mu}$, then*

$$\int_X \phi(x)\mu(dx) \leq \int_X \phi_1(x)\mu(dx). \tag{1}$$

Proof. For a non-negative martingale $\int_X |\phi_n(x)|\mu(dx) = \int_X \phi_n(x)\mu(dx) = \int_X \phi_1(x)\mu(dx)$, consequently, Condition 2.40(3) is satisfied. Then Inequality (1) follows from the Fatou Theorem II.6.2 [Shir89].

2.44. Corollary. *If $\{\phi_n : n\}$ is a non-negative sub-martingale and*

$$\sup_n \int_X \phi_n(x)\mu(dx) < \infty,$$

then the limit

$$\phi(x) = \lim_{n\to\infty} \phi_n(x) \pmod{\mu} \tag{1}$$

exists and

$$\int_X \phi(x)\mu(dx) \leq \sup_n \int_X \phi_n(x)\psi(x)\mu(dx). \tag{2}$$

If in (2) the equality is, then

$$\int_X \phi(x)\psi(x)\mu(dx) \leq \int_X \phi_n(x)\psi(x)\mu(dx) \tag{3}$$

for each $\mathcal{U}_n$-measurable bounded non-negative function ψ.

Proof. In view of the Fubini theorem (see Theorem 8 in §II.6 [Shir89]) we have

$$\int_X \phi(x)\psi(x)\mu(dx) \leq \lim_{m\to\infty} \int_X \phi_m(x)\psi(x)\mu(dx). \tag{4}$$

For each $b > \psi(x)$ the inequality

$$\int_X \phi(x)[b-\psi(x)]\mu(dx) \leq \lim_{m\to\infty} \int_X \phi_m(x)[b-\psi(x)]\mu(dx) \tag{5}$$

is satisfied. If in either (4) or (5) would be the strict inequality, then the inequality $\int_X \phi(x)\mu(dx) < \lim_{m\to\infty} \int_X \phi_m(x)\mu(dx)$ would be satisfied. This would contradict that in (2) is the equality. For martingales under the same conditions in (3) is the equality.

2.45. Note. If $\{\phi_n : n\}$ is a martingale and g is a non-negative convex (downward) function defined on $(-\infty,\infty)$ so that $g(\phi_n(x))\mu(dx) < \infty$ for each $n \in \mathbf{N}$, then $\{g(\phi_n(x)) : n\}$ is the sub-martingale. Indeed, in view of Remark 2.40 and the Jensen's inequality (see §II.6 [Shir89])

$$\int_X g(\phi_{n+1}(x))\psi(x)\mu(dx) = \int_X \left[\int_X g(\phi_{n+1}(y))\mu(dy, \mathcal{U}_n|x)\right]\psi(x)\mu(dx)$$

$$\geq \int_X g\left(\int_X \phi_{n+1}(y)\mu(dy,\mathcal{U}_n|x)\right)\psi(x)\mu(dx)$$

$$= \int_X g(\phi_n(x))\psi(x)\mu(dx) \quad (1)$$

for each non-negative $\mathcal{U}_n$-measurable bounded function ψ. If $g(t)$ is non-decreasing and $\{\phi_n(x) : n\}$ is a sub-martingale, then the relations above with the inequality $\geq$ instead of the latter equality $=$ imply that $\{g(\phi_n(x)) : n\}$ is the sub-martingale.

In the process of the proof of Theorem 2.42 we have demonstrated that

$$\mu(\{x \in X : k_n(x) \leq m\}) = \mu(\{x \in X : \sup_{n\leq m} g_n(x) > \beta_2 + (n-1)(\beta_2 - \beta_1)\})$$

$$\leq 2\left[\int_X \phi_m^+(x)\mu(dx) + 2|\beta_2|]/[\beta_2 + (n-1)(\beta_2 - \beta_1)\right]. \quad (2)$$

The set $\{x \in X : k_n(x) \leq m\}$ coincides with the set of those x for which the sequence $\phi_1(x),\dots,\phi_m(x)$ not less than n times intersects the segment $[\beta_1,\beta_2]$. That is for each n there exist a sequence $k_1 < k_2 < \dots < k_n \leq m$ with $\phi_{k_1}(x) \geq \beta_2$, $\phi_{k_2}(x) \leq \beta_1$, $\phi_{k_3}(x) \geq \beta_2$ and so on.

1.3. Quasi-invariant Measures

In this section after few preliminary statements there are given the definition of a quasi-invariant measure and the theorems about quasi-invariance of measures relative to transformations of a Banach space X over $\mathbf{K}$.

3.1. Let X be a Banach space over $\mathbf{K}$, $(L_n : n)$ be a sequence of subspaces, $cl(\bigcup_n L_n) = X$, $L_n \subset L_{n+1}$ for each n, μ^j be probability measures, $\mu^2 \ll \mu^1$, $(\mu^j_{L_n})$ be sequences of weak distributions, also let there exist derivatives $\rho_n(x) = \mu^2_{L_n}(dx)/\mu^1_{L_n}(dx)$ and the following limit

$$\rho(x) := \lim_{n\to\infty} \rho_n(x) \quad (1)$$

exists.

Theorem. *If μ^j are real-valued probability measures and in addition*

$$(i) \quad \int_X \rho(x)\mu^1(dx) = 1$$

with $\rho \in L^1(\mu^1)$, then this is equivalent to the following: there exists

$$(ii) \quad \rho(x) = \mu^2(dx)/\mu^1(dx) \quad (mod\ \mu^1).$$

Proof. Let $\mu^2 \ll \mu^1$ and $\pi(x) = d\mu^2(x)/d\mu^1(x)$. Denote by $\mu^k_n(*|x)$ the conditional measure of μ^k relative to the σ-algebra B^{L_n}. Therefore, for each B^{L_n}-measurable non-negative bounded function $\psi(x)$ there are satisfied the equalities:

$$\int_X \psi(x)\mu^2(dx) = \int_X \psi(x)\pi(x)\mu^1(dx)$$

$$= \int_X \psi(x) \left[\int_X \pi(y) \mu_n^1(dy|x) \right] \mu^1(dx).$$

On the other hand,

$$\int_X \psi(x)\mu^2(dx) = \int_X \rho_n(P_{L_n}x)\psi(x)\mu^1(dx), \text{ hence} \tag{2}$$

$$\rho_n(x) = \int_X \pi(y)\mu_n^1(dy|x).$$

We shall demonstrate that $\rho_n(x)$ is uniformly relative to n integrable by the measure μ^1. For this we have to show that for each $b > 0$ there exists β so that

$$\int_{\{x \in X:\ \rho_n(x) > \beta\}} \rho_n(x)\mu^1(dx) < b. \tag{3}$$

Define the function $g_\beta(t) = 0$ for $\beta > t$ and $g_\beta(t) = t - \beta$ for $\beta \le t$. Then Relation (3) is equivalent with

$$\int_X g_\beta(\rho_n(x))\mu^1(dx) + \beta\mu^1(\{x \in X:\ \rho_n(x) > \beta\}) < b. \tag{4}$$

From

$$g_\beta(\rho_n(x)) = g_\beta\left(\int_X \pi(y)\mu_n^1(dy|x)\right) \le \int_X g_\beta(\pi(y))\mu_n^1(dy|x)$$

we infer that

$$\int_X g_\beta(\rho_n(x))\mu^1(dx) \le \int_X g_\beta(\pi(y))\mu^1(dy). \text{ Moreover,}$$

$$\beta\mu^1(\{x \in X:\ \rho_n(x) > \beta\}) \le 2\int_X g_{\beta/2}(\rho_n(x))\mu^1(dx)$$

$$\le 2\int_X g_{\beta/2}(\pi(x))\mu^1(dx).$$

Therefore, Conditions $(3,4)$ will be satisfied as soon as the parameter β is chosen such that $\int_X g_{\beta/2}(\pi(x))\mu^1(dx) < b/3$.

From the uniform relative to n integrability of $\rho_n(x)$ by the measure μ it follows that in the equality $\int_X \rho_n(x)\mu^1(dx) = 1$ we can take the limit by $n \to \infty$ under the integral. This demonstrates (i).

For each $A \in B^{L_n}$ in the relations

$$\mu^2(A) = \lim_{n\to\infty} \int_X \chi_A(x)\rho_n(x)\mu^1(dx) = \int_X \chi_A(x)\rho(x)\mu^1(dx)$$

we can take the limit with n tending to the infinity under the integral. This proves (ii).

Suppose now that (i) is satisfied. Then from Corollary 2.44 we deduce that

$$\int_X \rho(x)\psi(x)\mu^1(dx) = \int_X \rho_n(x)\psi(x)\mu^1(dx) = \int_X \psi(x)\mu^2(dx)$$

for every B^{L_n}-measurable bounded non-negative function $\psi(x)$. Thus

$$\int_A \rho(x)\mu^1(dx) = \int_A \mu^2(dx) = \mu^2(A) \tag{5}$$

is satisfied for each $A \in B^{L_n}$. The class of all functions for which (5) is accomplished contains B^{L_n} and is monotone, consequently, it contains also the minimal σ-algebra containing each B^{L_n}, hence it contains the entire σ-algebra $Bf(X)$. From (5) for each $A \in Bf(X)$ it follows that μ^2 is absolutely continuous relative to μ^1, $\mu^2 \ll \mu^1$, and Formula (ii) is satisfied.

3.2. Theorem. *Probability measures $\mu^j : Bf(X) \to \mathbf{R}$, $j = 1,2$, for a Banach space X over $\mathbf{K}$ are orthogonal $\mu^1 \perp \mu^2$ if and only if $\rho(x) = 0$ $(mod\ \mu^1)$.*

Proof. If $A \in B^{L_n}$, then for each $n < m$ we have $\mu^2(A) = \int_A \rho_m(x)\mu^1(dx)$. In view of the Fatou theorem taking the limit with m tending to the infinity gives

$$\mu^2(A) \geq \int_A \rho(x)\mu^1(dx). \tag{1}$$

This spreads on all $A \in Bf(X)$.

Suppose now that $\mu^2 \perp \mu^1$, hence there exists a set A so that $\mu^2(A) = 0$ and $\mu^1(X \setminus A) = 0$. From (1) we infer that $\int_A \rho(x)\mu^1(dx) = 0$. Since $\mu^1(X \setminus A) = 0$, then $\rho(x) = 0$ $(mod\ \mu^1)$. This demonstrates the necessity.

Let now $\rho(x) = 0$ $(mod\ \mu^1)$. We shall show that $\mu^1 \perp \mu^2$. Suppose the contrary. So we can present μ^2 in the form $\mu^2 = \beta\nu^1 + (1-\beta)\nu^2$, where $\nu^1 \ll \mu^1$ and $\nu^2 \perp \mu^1$, $0 < \beta \leq 1$. Put $\rho^1_n(x) := d\nu^1_{L_n}(P_{L_n}x)/d\mu^1_{L_n}(P_{L_n}x)$. In accordance with Theorem 3.1 there exists the limit

$$\lim_{n\to\infty} \rho^1_n(x) = d\nu^1(x)/d\mu^1(x) \quad (mod\ \mu^1).$$

But $\beta\rho^1_n(x) \leq \rho_n(x)$, consequently, $\lim_{n\to\infty}\rho^1_n(x) \leq \lim_{n\to\infty}\rho_n(x)/b = 0$, that is, $d\nu^1(x)/d\mu^1(x) = 0$ $(mod\ \mu^1)$ contradicting the supposition about the absolute continuity of ν^1 relative to μ^1, hence $\mu^1 \perp \mu^2$.

3.2.1. Theorem. *The function $\rho(x)$ defined by Relation 3.1(2) is the density of an absolute continuous part of the measure μ^2 relative to the measure μ^1, so that Formula 3.1 (ii) is accomplished in all cases.*

Proof. Let $\mu^2 = \beta\nu^1 + (1-\beta)\nu^2$ with $\nu^1 \ll \mu^1$ and $\nu^2 \perp \mu^1$ and $0 \leq \beta < 1$. Denote by $\nu^1_{L_n}$ and $\nu^2_{L_n}$ projections of measures ν^1 and ν^2 on L_n, also $\rho^1_n(x) = d\nu^1_{L_n}(P_{L_n}x)/d\mu^1_{L_n}(P_{L_n}x)$, $\rho^2_n(x) = d\nu^2(P_{L_n}x)/d\mu^1_{L_n}(P_{L_n}x)$. Then $\rho_n(x) = \beta\rho^1_n(x) + (1-\beta)\rho^2_n(x)$. In accordance with Theorem 3.1 the limit $\lim_{n\to\infty}\rho^1_n(x) = d\nu^1(x)/d\mu^1(x)$ $(mod\ \mu^1)$ exists. By Theorem 3.2 $\lim_{n\to\infty}\rho^2_n(x) = 0$ $(mod\ \mu^1)$, hence $\lim_{n\to\infty}\rho_n(x) = \beta d\nu^1(x)/d\mu^1(x) = d\mu^2(x)/d\mu^1(x)$ $(mod\ \mu^1)$.

3.2.2. Theorem. *Let μ^1 and μ^2 be arbitrary Borel probability measures on a Banach space X over $\mathbf{K}$, let also L_n be an increasing sequence of $\mathbf{K}$-vector subspaces in X so that $\bigcup_n L_n$ is dense in X. Then*

$$d\mu^2(x)/d\mu^1(x) = \lim_{n\to\infty} d\mu^2_{L_n}(P_{L_n}x)/d\mu^1_{L_n}(P_{L_n}x) \quad (mod\ \mu^1), \tag{1}$$

while $\mu^2 \ll \mu^1$ if and only if $\int_X [d\mu^2(x)/d\mu^1(x)]\mu_1(dx) = 1$.

Proof. Construct the decomposition $\mu^2 = \nu^1 + \nu^2$ such that $\nu^2_{L_n} \ll \mu^1_{L_n}$ for each $n \in \mathbf{N}$ and

$$\lim_{n\to\infty}\int_X [d\nu^1_{L_n}(x)/d\mu^1_{L_n}(x)]\mu^1_{L_n}(dx) = 0. \tag{2}$$

Consider a set $A_n \in Bf(L_n)$ such that $\mu^1_{L_n}(A_n) = 0$ and the measure $\mu^2(B \setminus A_n)$ is absolutely continuous relative to $\mu^1_{L_n}(B)$ on $Bf(L_n)$. Denote by E_n the cylinder set in X with the base

A_n. Put $E = \bigcup_{n=1}^{\infty} E_n$, $\nu^2(B) = \mu^2(B \setminus E)$, $\nu^1(B) = \mu^2(B \cap E)$. Then $\nu^2_{L_n}(A) \le \mu^2_{L_n}(A \setminus A_n)$, consequently, $\nu^2_{L_n} \ll \mu^1_{L_n}$ by the construction of the set A_n. From $\mu^1_{L_n}(P^{-1}_{L_n}[A_k \cap L_n]) = 0$ for all $k \le n$ and from the fact that $d\nu^1_{L_n}/d\mu^1_{L_n}$ is different from zero only on the set $P_{L_n}E$ we get

$$\int_X [d\nu^1_{L_n}(x)/d\mu^1_{L_n}(x)]\mu^1_{L_n}(dx) = \nu^1_{L_n}\left(P_{L_n}\left[E \setminus \bigcup_{k=1}^{n} E_k\right]\right),$$

since $\int_A [d\nu(x)/d\mu(x)]\mu(dx) \le \nu(A)$ for any non-negative bounded measures.

Consider the cylinder set C_n from B^{L_n} with the base $P_{L_n}[E \setminus \bigcup_{k=1}^{n} A_k]$. Evidently, $C_n \supset C_{n+1}$ and $\lim_{n\to\infty} \nu^1(C_n) = \lim_{m\to\infty} \nu^1(\bigcap_{n=1}^{m} C_n)$. Since $E_n \cap C_n = \emptyset$, then $\bigcap_{n=1}^{\infty} C_n$ has the void intersection with each E_k and hence with $\bigcup_k E_k = E$ also. Thus $\nu^1(\bigcap_n C_n) = \nu^1(E \cap [\bigcap_n C_n]) = 0$. From the inequality $\int_X [d\nu^1(x)/d\mu^1(x)]\mu^1_{L_n}(dx) \le \nu^1(C_n)$ Equality (2) follows.

For the measure ν^2 the statements of Theorems 3.1-3.2.1 are accomplished. From the construction of ν^1 for each x in X outside the set $[\bigcap_n C_n] \cup E$ we have

$$\lim_{n\to\infty} d\nu^1_{L_n}(P_{L_n}x)/d\mu^1_{L_n}(P_{L_n}x) = 0. \tag{3}$$

The latter equality is satisfied μ^1-almost everywhere, since $\nu^1(\bigcap_n C_n) = 0$ and $\mu^1(E) = 0$. Therefore,

$$\lim_{n\to\infty} d\mu^2_{L_n}(P_{L_n}x)/d\mu^1_{L_n}(P_{L_n}x) = \lim_{n\to\infty} d\nu^2_{L_n}(P_{L_n}x)/d\mu^1_{L_n}(P_{L_n}x)$$
$$= d\nu^2(x)/d\mu^1(x) = d\mu^2(x)/d\mu^1(x) \quad (mod\ \mu^1).$$

3.2.3. Theorem. *Let the measurable Banach space $(X, \mathcal{B}(X))$, where X is over* **K** *and $(X, \mathcal{B}(X)) = (X_1, \mathcal{B}(X_1)) \times (X_2, \mathcal{B}(X_2))$. Suppose that two measures ν^j and μ^j are given on $(X_j, \mathcal{B}(X_j))$ for $j = 1, 2$. For measures $\mu = \mu^1 \times \mu^2$ and $\nu = \nu^1 \times \nu^2$ on $X, \mathcal{B}(X))$ the relation $\nu \ll \mu$ is satisfied if and only if $\nu^1 \ll \mu^1$ and $\nu^2 \ll \mu^2$. If this is the case, then*

$$d\nu(x)/d\mu(x) = [d\nu^1(P_1x)/d\mu^1(P_1x)][d\nu^2(P_2x)/d\mu^2(P_2x)], \tag{1}$$

where $P_1x = x_1 \in X_1$ and $P_2x = x_2 \in X_2$, $x = (x_1, x_2)$.

Proof. At first prove the necessity. If $\nu \ll \mu$ and $\mu(A_1 \times X_2) = \mu^1(A_1) = 0$, then $\nu(A_1 \times X_2) = \nu^1(A_1) = 0$ for each $A_1 \in \mathcal{B}(X_1)$. Therefore, $\nu^1 \ll \mu^1$. Analogously $\nu^2 \ll \mu^2$.

Let now $\nu^j \ll \mu^j$ for $j = 1, 2$. Denote by F^0 the algebra of sets from $\mathcal{B}(X)$ having the form $\bigcup_{k=1}^{n}(A^1_k \times A^2_k)$, where $A^j_k \in \mathcal{B}(X_j)$, $k, n \in \mathbf{N}$. Let $\rho(x) = [d\nu^1(P_1x)/d\mu^1(P_1x)][d\nu^2(P_2x)/d\mu^2(P_2x)]$. From

$$\int_{A_1\times A_2} \rho(x)\mu(dx) = \int_{A_1} [d\nu^1(x_1)/d\mu^1(x_1)]\mu^1(dx_1) \int_{A^2} [d\nu^2(x_2)/d\mu^2(x_2)]\mu^2(dx_2)$$
$$= \nu^1(A^1)\nu^2(A^2) = \nu(A^1 \times A^2)$$

for each $A \in F^0$ the equality

$$\int_A \rho(x)\mu(dx) = \nu(A) \tag{2}$$

follows. The latter relation is satisfied on a monotone class containing F^0, hence it is satisfied for all $A \in \mathcal{B}(X)$. This implies $\nu \ll \mu$ and Formula (1).

3.2.4. Theorem. *If probability measures μ^1 and μ^2 are defined on a measurable space $(X,\mathcal{B}(X))$, where X is a Banach space over* $\mathbf{K}$, $\mu^2 \ll \mu^1$,

$$\nu^j(C) := \mu^j(f^{-1}(C)),\ \forall C \in \mathcal{B}(Y)), \tag{1}$$

$f : (X,\mathcal{B}(X)) \to (Y,\mathcal{B}(Y))$ *is a measurable mapping, then*

$$d\nu^2(y)/d\nu^1(y) = \int_X [d\mu^2(x)/d\mu^1(x)]\mu^1(dx,\mathcal{B}_1|f^{-1}(y)), \tag{2}$$

where $\mu^1(*,\mathcal{B}_1|z)$ is the conditional measure of μ^1 relative to the σ-algebra $\mathcal{B}_1$ generated by sets of the form $f^{-1}(C), C \in \mathcal{B}(Y)$.

Proof. Show at first that $\mu^1(A,\mathcal{B}_1|z)$ for $A \in \mathcal{B}(X)$ is constant on the inverse image $f^{-1}(y)$ for any f. Consider $A_y := \{z : f(z) = y\}$, $A_y \in \mathcal{B}_1$. For each $\mathcal{B}_1$-measurable set A^1 either $A_y \subset A^1$ or $A_y = X \setminus A^1$. Therefore, each $\mathcal{B}_1$-measurable function is constant on A_y. The function $\mu^1(A,\mathcal{B}_1|z)$ of z is $\mathcal{B}_1$-measurable, hence $\mu^1(A,\mathcal{B}_1|f^{-1}(y))$ is independent from a choice of a point in the inverse image $f^{-1}(y)$ of a point y.

Let $\psi(y)$ be a bounded $\mathcal{B}(Y)$-measurable function on $(Y,\mathcal{B}(Y))$. Then the function $\psi(f(x)) = \phi(x)$ is $\mathcal{B}_1$-measurable and bounded on $(X,\mathcal{B}(X))$. Thus

$$\int_X \psi(y)\nu^2(dy) = \int_X \psi(f(x))\mu^2(dx) = \int_X \psi(f(x))[d\mu^2(x)/d\mu^1(x)]\mu^1(dx).$$

Expression 2.37(2) implies that

$$\int_X \psi(f(x))[d\mu^2(x)/d\mu^1(x)]\mu^1(dx) = \int_X \psi(f(z)) \int_X [d\mu^2(x)/d\mu^1(x)]\mu^1(dx,\mathcal{B}_1|z)\mu^1(dz).$$

Put

$$\int_X [d\mu^2(x)/d\mu^1(x)]\mu^1(dx,\mathcal{B}_1|f^{-1}(y)) = \rho^1(y), \tag{3}$$

then $\int_X \psi(f(x))\rho^1(f(z))\mu^1(dz) = \int_X \psi(y)\rho^1(y)\nu^1(dy)$, consequently, $\int_X \psi(y)\nu^2(dy) = \int_X \psi(y)\rho^1(y)\nu^1(dy)$, that demonstrates this theorem.

3.2.5. Corollary. *Let conditions of Theorem* 3.2.4 *be satisfied and in addition on $(X,\mathcal{B}(X))$ be two other measures ν^1 and ν^2 be given such that $\nu^j \ll \mu^j$, $j = 1,2$. Then $\nu^1 \times \nu^2 \ll \mu^1 \times \mu^2$ and $[d(\nu^1 \times \nu^2)/d(\mu^1 \times \mu^2)](x_1,x_2) = [d\nu^1/d\mu^1](x_1)[d\nu^2/d\mu^2](x_2)$. Moreover, $\nu^1 * \nu^2 \ll \mu^1 * \mu^2$ and*

$$[d(\nu^1 * \nu^2)/d(\mu^1 * \mu^2)](x) \tag{1}$$

$$= \int_X [d\nu^1/d\mu^1](x_1)[d\nu^2/d\mu^2](x_2)(\mu^1 * \mu^2)(dx_1 \times dx_2,\mathcal{B}^*|\eta^{-1}(x)),$$

where $\eta(x_1,x_2) = x_1 + x_2$, $\mathcal{B}^*$ denotes the σ-algebra generated by the sets $\eta^{-1}(A)$, $A \in \mathcal{B}(X)$.

Proof. As above the convolution of measures is $\mu^1 * \mu^2(A) = \int_X \mu^2(A-x)\mu^1(dx)$, where $A \in \mathcal{B}(X)$, $A - x := \{y \in X :\ y + x \in A\}$. The set $S^A = \{(x_1,x_2) : x_1 + x_2 \in A\}$ is $\mathcal{B}(X) \times \mathcal{B}(X)$-measurable for $\mathcal{B}$-measurable A. Put $S^A_{x_1} := \{x_2 \in X : (x_1,x_2) \in S^A\}$ is the section of S^A by the first coordinate. On the other hand,

$$[\mu^1 \times \mu^2](S^A) = \int_X \mu^2(S^A_{x_1})\mu^1(dx_1) = \int_X \mu^2(A - x_1)\mu^1(dx_1) \tag{2}$$

$$= \int_X \mu^1(S^A_{x_2})\mu^2(dx_2) = \int_X \mu^1(A - x_2)\mu^2(dx_2)$$

. This implies that the convolution is commutative. The convolution is obtained from the product of measures with the help of the mapping $\eta(x_1, x_2) = x_1 + x_2$. So this corollary follows from Theorems 3.2.3 and 3.2.4.

3.3. Note. For real-valued measures μ^j on $Bf(X)$ for a Banach space X over the infinite locally compact field $\mathbf{K}$ with the non-trivial multiplicative non-Archimedean norm there is the important Kakutani's theorem formulated in §3.3.1 below. Its proof is given in Theorem 4.1 §II.4.6[DF91] and in [Kak48] for abstract measurable spaces.

Let μ and ν be two probability measures in a measurable space $(X, \mathcal{B})$ and let λ be a probability measure in $(X, \mathcal{B})$ such that μ and ν are absolutely continuous relative to λ. Then the integral $H(\mu, \nu) := \int_X (d\mu/d\lambda)^{1/2}(d\nu/d\lambda)^{1/2} d\lambda$ is called the Hellinger integral of μ and ν. It has the properties:

(i) $0 \le H(\mu, \nu) \le 1$;

(ii) $H(\mu, \nu) = 1$ is equivalent to $\mu = \nu$;

(iii) $H(\mu, \nu) = 0$ if and only if μ is orthogonal to ν;

(iv) if μ is equivalent to ν, then $H(\mu, \nu) > 0$.

3.3.1. Theorem. *Let μ_n and ν_n be two sequences of probability measures in $(X_n, \mathcal{B}_n)$ (see above), define $\mu = \prod_{n=1}^\infty \mu_n$, $\nu = \prod_{n=1}^\infty \nu_n$ in $(X, \tilde{\mathcal{B}})$, where $\tilde{\mathcal{B}} := \bigotimes_{n=1}^\infty \mathcal{B}_n$. If μ_n is equivalent to ν_n for each $n \in \mathbf{N}$, then μ and ν are either equivalent or orthogonal depending on whether $\prod_{n=1}^\infty H(\mu_n, \nu_n)$ converges to a finite positive number or diverges to zero, where $d\mu/d\nu = \prod_{n=1}^\infty d\mu_n/d\nu_n$ when μ and ν are equivalent.*

3.4. Theorem. *Let ν, μ, ν_j, μ_j be probability measures with values in $\mathbf{R}$, $X = \prod_{j=1}^\infty X_j$ be a product of completely regular spaces X_j with the small inductive dimension $ind(X_j) = 0$. Then $\nu \ll \mu$ if and only if two conditions are satisfied:*

(a) *$\nu_j \ll \mu_j$ for each j and*

(b) *$\prod_{j=1}^\infty \beta_j$ converges to β, $\infty > \beta > 0$, where $\beta_j := \|(\rho_j)^{1/2}\|_{L^1(X_j, \mu_j)}$, $\rho_j(x) = \nu_j(dx)/\mu_j(dx)$.*

Proof. The necessity of (a) is evident. If $\nu \ll \mu$, then

$$d\nu(x)/d\mu(x) = \prod_{k=1}^\infty d\nu_k(P_k x)/d\mu_k(P_k x).$$

Therefore,

$$d\nu(x)/d\mu(x) = \lim_{n\to\infty} \prod_{k=1}^n d\nu_k(P_k x)/d\mu_k(P_k x). \qquad (1)$$

The functions $g_n(x) := \prod_{k=1}^n [d\nu_k(P_k x)/d\mu_k(P_k x)]^{1/2}$ are uniformly relative to n integrable, since the integrals $\int_X [g_n(x)]^2 \mu(dx) = 1$ are uniformly bounded. Combining this with (1) implies

$$\int_X [d\nu(x)/d\mu(x)]^{1/2}\mu(dx) = \lim_{n\to\infty} \int_X \prod_{k=1}^n [d\nu_k(P_k x)/d\mu_k(P_k x)]^{1/2}\mu(dx)$$

$$= \lim_{n\to\infty} \prod_{k=1}^n \int_X [d\nu_k(P_k x)/d\mu_k(P_k x)]^{1/2}\mu(dx),$$

hence (b) is proved.

Prove now the sufficiency of conditions of this theorem. In view of Theorem 3.3.1 under the condition (a) there is the alternative either $\nu \ll \mu$ or $\nu \perp \mu$. The latter will be, if $\sum_{k=1}^{\infty} \mu_k(B_k) = \infty$ or $\prod_{k=1}^{\infty} d\nu_k(P_k x)/d\mu_k(P_k x) = 0$ $(mod\ \mu)$. This means that it is sufficient to show that if $\nu \perp \mu$ and (a) is satisfied, then the infinite product in (b) diverges to zero. In this case $\lim_{n\to\infty} \prod_{k=1}^{\infty} d\nu_k(P_k x)/d\mu_k(P_k x) = 0$ $(mod\ \mu)$, consequently,

$$\lim_{n\to\infty} \prod_{k=1}^{n} [d\nu_k(P_k x)/d\mu_k(P_k x)]^{1/2} = 0. \tag{2}$$

Using the uniform integrability we infer that Equality (2) can be integrated interchanging the operations of integration and taking of the limit. Thus

$$\lim_{n\to\infty} \int_X \prod_{k=1}^{n} [d\nu_k(P_k x)/d\mu_k(P_k x)]^{1/2} \mu(dx) = \lim_{n\to\infty} \prod_{k=1}^{n} \int_X [d\nu_k(P_k x)/d\mu_k(P_k x)]^{1/2} \mu(dx).$$

3.5. Definition. Let X be a Banach space over $\mathbf{K}$, Y be a completely regular space with the small inductive dimension $ind(X) = 0$, $\nu : \mathcal{E} \to \mathbf{R}$, $\mu^y : \mathcal{B} \to \mathbf{R}$ for each $y \in Y$, suppose $\mu^y(A) \in L^1(Y, \nu, \mathbf{R})$ for each $A \in \mathcal{B}$, where $\mathcal{B}$ is a σ-algebra on X and $\mathcal{E}$ is a σ-algebra on Y. Then we define:

$$(i)\ \mu(A) = \int_Y \mu^y(A)\nu(dy).$$

A measure μ is called mixed. We define measures π^j by the formula:

$$(ii)\ \pi^j(A \times C) = \int_C \mu^{j,y}(A)\nu^j(dy),$$

where $j = 1, 2$ and $\mu^{y,j}$ together with ν^j are defined as above μ^y and ν.

3.5.1. Theorem. *If π^j, $\mu^{j,y}$ and ν^j are defined as in §3.5, $j = 1, 2$, X is of separable type over the locally compact field $\mathbf{K}$, where $\mu^{j,y}$ and ν^j are bounded non-negative measures with $0 < \nu^j(Y) < \infty$ and $0 < \mu^{j,y}(X) < \infty$ for each j and y, then*

(i) if $\pi^2 \ll \pi^1$, then $\nu^2 \ll \nu^1$ and $\mu^{2,y} \ll \mu^{1,y}$ for ν^2-almost all y;

(ii) if $\nu^2 \ll \nu^1$ and $\mu^{2,y} \ll \mu^{1,y}$ for ν^2-almost all y, then $\pi^2 \ll \pi^1$. Moreover, there exists a $\mathcal{B} \times \mathcal{E}$-measurable function $\eta(y,x) = d\mu^{2,y}(x)/d\mu^{1,y}(x)$ so that

$$d\pi^2(x,y)/d\pi^1(x,y) = [d\nu^2(y)/d\nu^1(y)]\eta(y,x). \tag{1}$$

Proof. Let $\pi^2 \ll \pi^1$. We put $\rho(y,x) = d\pi^2(x,y)/d\pi^1(x,y)$. For each $\mathcal{B} times \mathcal{E}$-measurable bounded function $\phi(x,y)$ the function $\int_X \phi(x,y)\mu^{k,y}(dx)$ is $\mathcal{E}$-measurable and

$$\int_{X\times Y} \phi(x,y)\pi^j(dx,dy) = \int_Y \left[\int_X \phi(x,y)\mu^{j,y}(dx)\right]\nu^j(dy). \tag{2}$$

Therefore, for each $B \in \mathcal{B}$ and $C \in \mathcal{E}$ this implies that

$$\pi^2(B \times C) = \int_C \int_B \rho(y,x)\mu^{1,y}(dx)\nu^1(dy)$$

$$= \int_C \int_B \left[\rho(y,x)\mu^{1,y}(dx)/\int_X \rho(y,z)\mu^{1,y}(dz)\right]\left[\int_X \rho(y,z)\mu^{1,y}(dz)\right]\nu^1(dy).$$

Taking $B = X$ we get

$$\pi^2(X \times C) = \nu^2(C) = \int_C \left[\int_X \rho(y,x)\mu^{1,y}(dx) \right] \nu^1(dy,$$

hence $\nu^2 \ll \nu^1$ and

$$d\nu^2(y)/d\nu^1(y) = \int_X \rho(y,x)\mu^{1,y}(dx)$$

and inevitably

$$\pi^2(B \times C) = \int_C \mu^{2,y}(B)\nu^2(dy) = \int_C \left[\int_B \eta(y,x)\mu^{1,y}(dx) \right] \nu^2(dy),$$

where

$$\eta(y,x) = \rho(y,x) \left[\int_X \rho(y,x)\mu^{1,y}(dx) \right]^{-1}. \tag{3}$$

Thus

$$\mu^{2,y}(B) = \int_B \eta(y,x)\mu^{1,y}(dx) \tag{4}$$

for ν^2-almost all y and each $B \in \mathcal{B}$. Choose a sequence of sets $B_k \in \mathcal{B}$ such that it will form an algebra generating $\mathcal{B}$. This is possible, since the field **K** is separable and X is of separable type over X with a countable base of neighborhoods of zero. Therefore, there exists a set $F \in \mathcal{E}$ so that $\nu^2(F) = \nu^2(Y)$ and

$$\mu^{2,y}(B_k) = \int_{B_k} \eta(y,x)\nu^{1,y}(dx)$$

for each $y \in F$ and all k, consequently, (4) is satisfied for all $y \in F$ and $B \in \mathcal{B}$.

For proving (ii) we first show that $\pi^2 \ll \pi^1$. Consider $A \in \mathcal{B} \times \mathcal{E}$, $A_y = \{x : (x,y) \in A\}$, so from (2) it follows that $\pi^j(A) = \int_Y \mu^{j,y}(A_y)\nu^j(dy)$. If $\pi^1(A) = 0$, then $\mu^{1,y}(A_y) = 0$ $(mod\ \nu^1)$, hence $\mu^{2,y}(A_y) = 0$ $(mod\ \nu^2)$, since $\nu^2 \ll \nu^1$ and $\mu^{2,y} \ll \mu^P 1,y$ for ν^2-almost all y. Thus $\pi^2(A) = 0$, that is $\pi^2 \ll \pi^1$.

For proving the existence of η and demonstrating (1) use the proof of (i) defining η by Formula (3).

3.5.2 Corollary. *If conditions of the Theorem* 3.5.1 *are satisfied and* $\nu^2 \ll \nu^1$ *and* $\mu^{2,y} \ll \mu^{1,y}$ *for* ν^2*-almost all* y*, then* $\mu^2 \ll \mu^1$ *and*

$$d\mu^2(x)/d\mu^1(x) = \int_X [d\nu^2(y)/d\nu^1(y)][d\mu^{2,y}(x)/d\mu^{1,y}(x)]\pi^1(dy|x), \tag{1}$$

where $\pi^1(C|x)$ *is the conditional measure defined by the equation:*

$$\int_B \pi^1(C|x)\mu^1(dx) = \pi^1(B \times C). \tag{2}$$

Proof. This follows from Theorems 3.2.4 and 3.5.1. Formula (1) is the consequence of 3.2.4(2), since $f((x;y)) = x$ in the considered situation.

3.5.3 Lemma. *Let conditions of Theorem* 3.5.1 *be satisfied. If in addition*

(i) *there exists a measure* λ *such that* $\mu^{1,y} \ll \lambda$ *for all* y*, then*

$$d\mu^2(y)/d\mu^1(y) = \int_Y \eta(y,x)[d\mu^{1,y}(x)/d\lambda(x)]\nu^2(dy)/\int_Y [d\mu^{1,y}(x)/d\lambda(x)]\nu^1(dy); \qquad (1)$$

(ii) *all measures* $\mu^{1,y}$ *for different* y *are orthogonal to each other for different values of* y*, moreover, there exist such pairwise non-intersecting* $B_y \in \mathcal{B}$ *so that* $\mu^{1,y}(B_y) = 1$ *and* $\mu^{1,y}(B_z) = 0$ *for* $z \neq y$ *and the function* $\rho(x) = d\mu^{2,y}(x)/d\mu^{1,y}(x)$ *with* $x \in B_y$ *is* $\mathcal{B}$*-measurable, also* $\nu^1 = \nu^2$*, then* $d\mu^2(x)/d\mu^1(x) = \rho(x)$*.*

Proof. (i). Let $\zeta(x,y) = d\mu^{1,y}(x)/d\lambda(x)$. We can choose $\zeta(x,y)$ to be $\mathcal{B}\times\mathcal{E}$-measurable using Theorem 3.5.1 and considering besides the measure π^1 also the measure $u = \lambda\times\nu^1$. Then

$$\pi^1(B\times C) = \int_B\int_C \zeta(x,y)\lambda(dx)\nu^1(dy),$$

$$\mu^1(B) = \pi^1(B\times Y) = \int_B\int_Y \zeta(x,y)\nu^1(dy)\lambda(dx),$$

consequently, $\mu^1 \ll \lambda$ and

$$d\mu^1(x)/d\lambda(x) = \int_Y \zeta(x,y)\nu^1(dy). \qquad (2)$$

Evidently, $\mu^{2,y} \ll \lambda$ and

$$d\mu^{2,y}(x)/d\mu^{1,y}(x) = \eta(y,x)\zeta(x,y).$$

We find also

$$d\mu^2(x)/d\lambda(x) = \int_Y \eta(y,x)\zeta(x,y)\nu^2(dy). \qquad (3)$$

From $\mu^2 \ll \mu^1 \ll \lambda$ we deduce that

$$d\mu^2/d\mu^1 = [d\mu^2/d\lambda][d\mu^1/d\lambda]^{-1},$$

hence from $(2,3)$ Equation (1) follows.

To demonstrate (ii) mention that

$$\int_X \phi(x)\rho(x)\mu^1(dx) = \int_Y\left[\int_X \phi(x)\rho(x)\mu^{1,y}(dx)\right]\nu^1(dy)$$

$$= \int_Y\left[\int_X \phi(x)[d\mu^{2,y}(x)/d\mu^1 1,y(x)]\mu^{1,y}(dx)\right]\nu^1(dy)$$

$$= \int_Y\left[\int_X \phi(x)\mu^{2,y}(dx)\right]\nu^1(dy) = \int_X \phi(x)\mu^2(dx)$$

for each bounded $\mathcal{B}$-measurable function $\phi(x)$.

3.5.4. Theorem. *Let the families of measures* $\mu^{j,y}$*,* $j = 1,2$*, on* $(X,\mathcal{B})$ *satisfy conditions:*

(i) $\mu^{j,y}(B)$ *is* $\mathcal{E}$*-measurable for each* $B \in \mathcal{E}$*;*

(ii) *there exist such* $\mathcal{B}$*-measurable sets* B_y *so that* $\mu^{j,y}(B_z) = 1$ *for* $y = z$ *and* $\mu^{j,y}(B_z) = 0$ *for* $z \neq y$ *and* $\bigcup_{y\in C} B_y \in \mathcal{B}$ *for each* $C \in \mathcal{E}$*,* $B_y \cap B_z = \emptyset$ *for each* $y \neq z$*, while* ν^j *is a measure* *on* $(Y,\mathcal{E})$ *for* $j=1,2$. *Suppose that measures* μ^j *are defined by Equation* 3.5(i) *with* $\mu^{j,y}$ *and* ν^j *instead of* μ^y *and* ν *respectively, where* $\mu^{j,y}$ *and* ν^j *are bounded non-negative measures with* $0<\nu^j(Y)<\infty$ *and* $0<\mu^{j,y}(X)<\infty$ *for each* j *and* y. *Then* $\mu^2 \ll \mu^1$ *if and only if:* (iii) $\nu^2 \ll \nu^1$ *and* (iv) $\mu^{2,y} \ll \mu^{1,y}$ *for* ν^2*-almost all* y, *if so, then*

$$d\mu^2(x)/d\mu^1(x) = [d\mu^{2,y}(x)/d\mu^{1,y}(x)][d\nu^2(x)/d\nu^1(x)] \tag{1}$$

for each $x \in B_y$.

Proof. The mapping $f((x;y)) = x$ admits almost everywhere by the measure inversion $(x;y) = g(x)$, where $g(x) = (x;y)$ for $x \in B_y$. Thus $g(x)$ is defined on $G := G(C) := \bigcup_{y\in C} B_y$, where $C \in \mathcal{E}$. Consider the algebra Υ which is the intersection $\bigcap_y \mathcal{B}_{\mu^{1,y}}$ of completions $\mathcal{B}_{\mu^{1,y}}$ of the algebra $\mathcal{B}$ by the measures $\mu^{1,y}$. Then $G \in \Upsilon$. If λ^1 is the completion of the measure μ^1, then G is λ^1-measurable and $\lambda^1(G) = 1$.

Choose B_y so that $G(C) \in \mathcal{B}$ for each $C \in \mathcal{E}$. Then $g(x)$ is $\mathcal{B}$-measurable and $\{x : g(x) \in B\times C\} = (G(C)\cap B \in \mathcal{B}$. Consider as well the measure v on $\mathcal{B}\times\mathcal{E}$ which is the image of the measure μ^1 under the mapping g such that

$$v(B\times C) = \mu^1(G(C)\cap B) = \int_Y \mu^{1,z}(G(C)\cap B)\nu^1(dz)$$

$$= \int_Y \mu^{1,z}(B_z\cap B)\chi_C(z)\nu^1(dz) = \int_Y \mu^{1,z}(B)\chi_C(z)\nu^1(dz) = \pi^1(B\times C),$$

since $\mu^{1,z}(G(C)\cap B) = \mu^{1,z}(B_z\cap[G(C)\cap B]) = \chi_C(y)\mu^{1,z}(B)$, where $\chi_C(y)$ is the characteristic function of the set C.

Prove now Formula (1). For this introduce the measure $w^1(B) := \int_Y \mu^{1,y}(B)\nu^2(dy)$. From Statement (ii) of Lemma 3.5.3 we know that

$$d\mu^2(x)/d\mu^1(x) = d\mu^{2,y}(x)/d\mu^{1,y}(x) \quad \text{for each } x \in B_y.$$

For each bounded $\mathcal{B}$-measurable function ϕ with $\rho(x) = d\nu^2(x)/d\nu^1(x)$ for each $x \in B_y$ there are the equalities

$$\int_X \phi(x)\mu^1(dx) = \int_Y \left[\int_X \phi(x)\mu^{1,y}(dx)\right]\nu^2(dy)$$

$$= \int_Y \left[\int_X \phi(x)\mu^{1,y}(dx)\right][d\nu^2(y)/d\nu^1(y)]\nu^1(dy) = \int_Y \left[\int_X \phi(x)\rho(x)\mu^{1,y}(dx)\right]\nu^1(dy)$$

$$= \int_X \phi(x)\rho(x)\mu^1(dx),$$

hence $dw^1(x)/d\mu^1(x) = \rho(x)$. Finally applying $d\mu^2/d\mu^1 = [d\mu^2/dw^1][dw^1/d\mu^1]$ implies (1).

3.6. Definition. For a Banach space X over a locally compact infinite field $\mathbf{K}$ with a non-trivial non-Archimedean multiplicative norm an element $a \in X$ is called an admissible shift of a measure μ, if $\mu_a \ll \mu$, where $\mu_a(A) = \mu(S_{-a}A)$ for each A in $Bf(X)$, $S_aA := a+A$, $\rho(a,x) := \rho_\mu(a,x) := \mu_a(dx)/\mu(dx)$, $M_\mu := [a \in X : \mu_a \ll \mu]$ (see §§2.1 and 2.36).

A vector $a \in X$ is called a partially admissible shift for the measure μ, if $d\mu_a/d\mu$ is not identical to zero relative to the measure μ, that is μ_a contains a component absolutely continuous relative to μ. In such situation denote $\tilde{\rho}_\mu(a,x) = d\mu_a(x)/d\mu(x)$ the density of the absolutely continuous component of μ_a relative to μ, the set of all partially admissible shifts we denote by $\tilde{M}_\mu$.

The proofs of Properties I-IV given below in §3.7 differ slightly from the proofs in Chapter II for $\mathbf{K_s}$-valued measures.

3.7. Properties of M_μ and ρ from § 3.6.

I. *The set M_μ is a semigroup by addition, $\rho(a+b,x) = \rho(a,x)\rho(b,x-a)$ for each $a,b \in M_\mu$.*

II. *If $a \in M_\mu$, $\rho(a,x) \neq 0$ $(mod\ \mu)$, then $\mu_a \sim \mu$, $-a \in M_\mu$ and $\rho(-a,x) = 1/\rho(a,x-a)$.*

III. *If $\nu \ll \mu$ and $\nu(dx)/\mu(dx) = g(x)$, then $M_\mu \cap M_\nu = M_\mu \cap [a : \mu([x : g(x) = 0,\ g(x-a)\rho_\mu(a,x) \neq 0]) = 0]$ and $\rho_\nu(a,x) = [g(x-a)/g(x)]\rho_\mu(a,x)$ $(mod\ \nu)$ for $a \in M_\mu \cap M_\nu$.*

IV. *If $\nu \sim \mu$, then $M_\nu = M_\mu$.*

V. *Let $\nu \ll \mu$ and $d\nu(x)/d\mu(x) = g(x)$, $a \in M_\mu$ only when $a \in \tilde{M}_\mu$,*

$$\rho_\nu(a,x) = g(x-a)\tilde{\rho}_\mu(a,x)/g(x) \quad (mod\ \nu). \tag{1}$$

Proof. Evidently $M_\mu \subset \tilde{M}_\mu$ and for $a \in M_\mu$ functions ρ_μ and $\tilde{\rho}_\mu$ coincide. If a is not an admissible shift for μ, then it is not admissible for ν, hence $M_\nu \subset \tilde{M}_\mu$. Then

$$\int_X f(x)\nu_a(dx) = \int_X f(x)\rho_\nu(a,x)\nu(dx)$$

$$= \int_X f(x)\rho_\nu(a,x)g(x)\mu(dx),$$

also

$$\int_X f(x+a)\nu(dx) = \int_X f(x+a)g(x)\mu(dx) = \int_X f(x)g(x-a)\mu_a(dx)$$
$$= \int_X f(x)g(x-a)\tilde{\rho}(a,x)\mu(dx) + \int_X f(x)g(x-a)\lambda_a(dx),$$

where $\lambda_a \perp \mu$, consequently, $\nu \perp \lambda_a$ and $\nu_a \perp \lambda_a$. If $f = 0$ $(mod\ \lambda_a)$, then from the equality

$$\int_X f(x)\rho_\nu(a,x)g(x)\mu(dx) = \int_X f(x)g(x-a)\tilde{\rho}(a,x)\mu(dx)$$

that demonstrates Formula (1).

VI. *Let $(X,\mathcal{B})$ and $(Y,\mathcal{E})$ be two Banach spaces over $\mathbf{K}$, $\mu : \mathcal{B} \to [0,\infty)$ be a bounded measure and $T : X \to Y$ be a $\mathbf{K}$-linear mapping. Denote by $\nu(C) := \mu(T^{-1}C)$ the measure on $(Y,\mathcal{E})$. If $a \in M_\mu$, then $b = Ta \in M_\nu$ and $\rho_\nu(b,y) = \int_X \rho_\mu(a,x)\mu(dx,\mathcal{B}_0|T^{-1}y)$, where $\mu(*,\mathcal{B}_0|y)$ is the conditional distribution of the measure μ relative to the σ-algebra $\mathcal{B}_0$ generated by sets $T^{-1}C, C \in \mathcal{E}$. In particular, if T is invertible, then $\rho_\nu(b,y) = \rho_\mu(T^{-1}b,T^{-1}y)$.*

Proof. This statement follows from Theorem 3.2.4, since each bounded non-zero measure μ induces the probability measure $\mu(A)/\mu(X)$, $A \in \mathcal{B}$.

VII. *Let X be a Banach space over $\mathbf{K}$ and on $(X,\mathcal{B})$ be given two non-negative bounded measures μ_1 and μ_2; $\mu = \mu_1 * \mu_2$ be the convolution of these measures. Then $a \in M_\mu$ for $a \in M_{\mu_1}$, moreover,*

$$\rho_\mu(a,x) = \int_X \rho_{\mu_1}(a,y)\mu(dy,\mathcal{B}_0|U^{-1}x), \tag{1}$$

where $\mu(dy, \mathcal{B}_0|x)$ *is the conditional measure corresponding to* $\mu_1 \times \mu_2$ *relative to the* σ*-algebra* $\mathcal{B}_0 \subset \mathcal{B} \times \mathcal{B}$ *generated by all sets of the form* $\{(x_1,x_2) : x_1 + x_2 \in A\}$, $a \in \mathcal{B}$, $U : X \times X \to X$ *so that* $U(x_1,x_2) := x_1 + x_2$.

Proof. This property also follows from Theorem 3.2.4 and also from Corollary 3.2.5.

VIII. *Let* μ *be a bounded non-negative measure on* $(X,\mathcal{B})$*. Define in* M_μ *the distance function*

$$r(a_1,a_2) := \int_X |\rho_\mu(a_1,x) - \rho_\mu(a_2,x)|\mu(dx). \tag{1}$$

Then (M_μ, r) *is the complete pseudo-metric space.*

Proof. Suppose that $a_n \in M_\mu$ and $\lim_{n\to\infty, m\to\infty} r(a_n,a_m) = 0$. We show that there exists $a \in M_\mu$ for which $\lim_{n\to\infty} r(a_n,a) = 0$.

Mention that the sequence $\{a_n : n\}$ is bounded in $(X, \|*\|)$. If $\|a_{n_k}\| \to \infty$ with $k \to \infty$, choose n_k such that $r(a_{n_k}, a_{n_{k+1}}) < 3^{-k}$. For $\mathcal{B}$-measurable function with $\|f\|_{L^1(\mu)} = 1$ and with a bounded support we obtain

$$\int_X f(x + a_{n_N})\mu(dx) \geq \int_X f(x+a_n)\mu(dx) - \sum_{k=1}^{N-1} r(a_{n_k}, a_{n_{k+1}}).$$

Choosing f so that $\int_X f(x+a_{n_1})\mu(dx) > 1/2$ and taking the limit with $N \to \infty$ we infer that $\lim_{N\to\infty} \int_X f(x+a_{n_N})\mu(dx) > 0$, that is impossible for f with the bounded support, when $\|a_{n_N}\| \to \infty$. This means that $\{a_n : n\}$ is bounded in X relative to the norm $\|*\|$ in the Banach space X.

Therefore, we can consider without loss of generality, that $\{a_n : n\}$ weakly converges to some a, since the topologically dual space denoted by X^* or by X' separates points in X for the locally compact field $\mathbf{K}$ (see also [NB85, Roo78] and the beginning of this chapter). From the relations

$$\int_X \chi_e(z(x+a_n))\mu(dx) = \chi_e(z(a_n)) \int_X \chi_e(z(x))\mu(dx) = \int_X \chi_e(z(x))\rho_\mu(a_n,x)\mu(dx),$$

taking into account that $\lim_{n\to\infty} z(a_n) = z(a)$ for each $z \in X^*$. There exists the limit $\lim_{n\to\infty} \rho_\mu(a_n,x) = \rho(x)$ by the measure μ and $\lim_{n\to\infty} \int_X |\rho_\mu(a_n,x) - \rho(x)|\mu(dx) = 0$ we find that $\chi_e(z(a)) \int_X \chi_e(z(x))\mu(dx) = \int_X \chi_e(z(x))\rho(x)\mu(dx)$. From this we find that $\int_X \chi_e(z(x))\mu_a(dx) = \int_X \chi_e(z(x))\rho(x)\mu(dx)$ for each $z \in X^*$. Thus $\mu_a \ll \mu$ and $\rho(x) = d\mu_a(x)/d\mu(x)$.

3.8. Definition and notes. A linear operator U on a Banach space X over $\mathbf{K}$ has a matrix representation $u(i,j) = \tilde{e}_i(Ue_j)$, where $(e_j : j)$ is a non-Archimedean orthonormal basis in X, $\tilde{e}_i$ are vectors of the topologically dual space X^* under the natural embedding (see §§ 2.2, 2.6). For a compact operator $U - I$ we get that there exists $\lim_{n\to\infty} det\{\hat{r}_n(U)\} \in \mathbf{K}$ which we adopt as the definition of $det\{U\}$, where $r_n : X \to \mathbf{K^n}$ is the projection operator with $\hat{r}_n(U) := \{u(i,j) : i,j = 1,\dots,n\}$ (see also Appendix, use that $\lim_{i+j\to\infty} u(i,j) = 0$).

3.9. Theorem. *Let* X *be a Banach space over* $\mathbf{K}$, μ *be a probability real-valued measure and* $T : X \to X$ *be a compact operator such that* $Re(1 - \hat{\mu}(z)) \to 0$ *for* $|\tilde{z}(T^2 z)| \to 0$ *and* $z \in X$, $\tilde{z} \in X'$ *corresponding to z. Then* $M_\mu \subset (TX)^\sim$*, where* $Y^\sim$ *is a completion by* $\|*\|_Y$ *of a normed space* Y.

Proof. For each $a \in M_\mu$ and $b > 0$ we have $J(z) = \int_X [1 - [\chi_e(\tilde{z}(x)) + \chi_e(-\tilde{z}(x))]/2]\rho(a,x)\mu(dx) \leq J_1(z) + J_2(z)$, where $J_1(z) := b\int_X [1 - [\chi_e(\tilde{z}(x)) +$

$\chi_e(-\tilde{z}(x))]/2]\mu(dx)$, $J_2(z) = 2\int_{(x:\ \rho(a,x)>b)} \mu(dx)$, $J_1(z) \to 0$ for $|\tilde{z}(T^2z)| \to 0$, $J_2(z) \to 0$ for $b \to +\infty$, consequently, $\lim_{|\tilde{z}(T^2z)|\to\infty} Re(\mu_a(X) - \chi_e(\tilde{z}(a))\hat{\mu}(z)) = 0$. From $|1-\hat{\mu}(z)|^2 \le 2Re(1-\hat{\mu}(z))$ and $\mu(X) = \mu_a(X) = 1$ it follows that $\chi_e(\tilde{z}(a)) \to 1$ and $\tilde{z}(a) \to 0$ for $|\tilde{z}(T^2z)| \to 0$. If $|\tilde{z}(a)| < \varepsilon$ for $|\tilde{z}(T^2z)| < \delta$, then $|\tilde{z}(a)|^2 < \varepsilon^2|\tilde{z}(T^2z)|/\delta$. Let $(e_j : j)$ be orthonormal basis in X such that $T = (T_{i,j} : i,j \in \mathbf{N})$, $\lim_{i+j\to\infty} T_{i,j} = 0$, we denote $D_j := \sup_k |T_{k,j}|$, hence $|a_j|^2 < \varepsilon^2 D_j^2/\delta$, $a \in (TX)^{\sim}$, $\|a\|_{TX} := \sup_j D_j|a_j| < \infty$ (see Appendix).

3.10. Corollary. *Let X and μ be the same as in* § 3.9, *then $L \cap M_\mu$ is a set of the first category in X for each infinite-dimensional linear subspace $L \subset X$ over* $\mathbf{K}$.

Proof follows from the fact that $(TB(X,0,p^j))^{\sim} =: A_j$ is nowhere dense in $L \cap B(X,0,p^j)$ for each $j \in \mathbf{N}$, since A_j is compact in X [Sch89].

The proof of Theorem 2 § 19[Sko74] can not be transferred on non-Archimedean X, there exists a compact V in $X = c_0(\omega_0,\mathbf{K})$ and its linear span $Y = span_{\mathbf{K}}V$ such that an ultranorm $p_E(x) := \inf[|a| : x \in aE]$ (see Exer. 6.204 and 5.202[NB85]) produces from Y non-separable and non-Radonian Banach space $l^\infty(\omega_0,\mathbf{K})$ [DF91, Roo78], where $E = co(V)$ in Y.

Note. Moreover, as it will be seen below from the proofs of Theorems 3.20 and 4.2 it follows the existence of a probability quasi-invariant measure $\mu : Bf(X) \to \mathbf{R}$ with $M_\mu \supset H + G_T$, where G_T is a compact subgroup in X with $\mu(G_T) > 0$ such that $\mu(L) = 0$ for real-valued μ and each linear subspace L in X with $dim_{\mathbf{K}}L < \aleph_0$ (see Chapter II also respectively).

3.11. Definition. For a Banach space X over $\mathbf{K}$ and a measure $\mu : Bf(X) \to \mathbf{R}$, $a \in X$, $\|a\| = 1$, a vector a is called an admissible direction, if $a \in M_\mu^{\mathbf{K}} := [z : \|z\|_X = 1, \lambda z \in M_\mu$ and $\rho(\lambda z, x) \ne 0$ $(mod\ \mu)$ (relative to x) and for each $\lambda \in \mathbf{K}] \subset X$. Let $a \in M_\mu^{\mathbf{K}}$ we denote by $L_1 := \mathbf{K}a$, $X_1 = X \ominus L_1$, μ^1 and $\tilde{\mu}^1$ are the projections of μ onto L_1 and X_1 respectively, $\tilde{\mu} = \mu^1 \otimes \tilde{\mu}^1$ be a measure on $Bf(X)$, given by the the following equation $\tilde{\mu}(A \times C) = \mu^1(A)\tilde{\mu}^1(C)$ on $Bf(L_1) \times Bf(X_1)$ and extended on $Bf(X)$, where $A \in Bf(L_1)$, $C \in Bf(X_1)$.

3.12. Theorem. *For a to be in $M_\mu^{\mathbf{K}}$ with a real-valued probability measure μ it is necessary and sufficient that the following conditions be satisfied (i-iv):*

(i) $\mu \ll \tilde{\mu}$;

(ii) $\mu^1 \ll m$, *where m is a real-valued Haar measure on L_1;*

(iii) $\mu^1(dx)/\mu(dx) := h(x) \ne 0$ *for* $|a(x)| > 0$;

(iv) $\mu([x : g(x-\lambda a) \ne 0,\ g(x) = 0]) = 0$ *for* $g(x) = \mu(dx)/\tilde{\mu}(dx)$ *whilst* $|\lambda| > 0$.

Proof. From $\tilde{\mu}_{\lambda a} = \mu^1_{\lambda a} \otimes \tilde{\mu}^1$ for $\lambda > 0$ and $\mu^1_{\lambda a} \ll \mu^1$, hence $\lambda a \in M_{\tilde{\mu}}$. In view of (iv) and and §§ 3.10 II, III the sufficiency follows, $\lambda a \in M_{\tilde{\mu}} \cap M_\mu$.

Now prove the necessity of Conditions $(i-iv)$. At first we demonstrate that $\mu \ll \tilde{\mu}$.

Recall the following proposition about measures equivalent with Haar measures. If G is a locally compact group and μ is a non-negative non-zero left invariant Haar measure on G, then a measure $\nu \ne 0$ on G is left quasi-invariant if and only if it is equivalent with μ (see Proposition 11 §VII.1.9[Bou63-69]).

From $\rho(\lambda z, x) \ne 0$ $(mod\ \mu)$ it follows that $\mu^1 \sim m$ due to the cited just above proposition, where m is the Haar measure on L_1. For each $A \in \mathcal{B}$ for $x \in L_1 = a\mathbf{K}$ denote by A^1_x the set of all $y \in X^1$ for which $(x,y) \in A$, where $X = L_1 \oplus X_1$, $L_1 \cap X_1 = \{0\}$, X_1 is the closed $\mathbf{K}$-linear subspace in X. Then $A^1_x \in Bf(X_1)$ and

$$\mu(A) = \int_X \mu(x, A^1_x)\mu^1(dx), \tag{1}$$

$$\tilde{\mu}(A) = \int_X \tilde{\mu}^1(A^1_x)\mu^1(dx). \tag{2}$$

Therefore, for proving $\mu \ll \tilde{\mu}$ it is sufficient to show that $\mu(x,*) \ll \tilde{\mu}^1$ for each x. Since $\tilde{\mu}^1(B_1) = \int_X \mu(x,B_1)\mu^1(dx)$, then form $\tilde{\mu}^1(B_1) = 0$ it follows that $\mu(x,B_1) = 0$ for μ^1-almost all x.

The density of μ^1 relative to the Haar measure m on L_1 at a point x for sufficiently large $\|(a,x)\|$ is positive, where m is positive on a unit ball in L_1. Then there exist λ with sufficiently large $|\lambda|$ for which $\mu(\lambda a, B_1) = 0$.

Using $\mu(\lambda_1 a, *) \ll \mu(\lambda a, *)$ for $|\lambda_1| < |\lambda|$ we get that $\mu(\lambda a, B_1) = 0$ for all λ. Hence from $\tilde{\mu}(B_1) = 0$ it follows that $\mu(x,B_1) = 0$ for each $x \in L_1$, that is $\mu(x,*) \ll \tilde{\mu}^1$.

Calculate now $d\mu(x)/d\tilde{\mu}(x) = g(x)$. In view of Theorem 3.5.1 and Formulas $(1,2)$ we deduce that $g(x) = d\mu(P_{L_1}x, P_{X_1}x)/d\tilde{\mu}(P_{X_1}x)$. Since $\mu(P_{L_1}x - \lambda a, *) \ll \mu(P_{L_1}x, *)$ for $\lambda \neq 0$, then

$$g(x-\lambda a) = g(x)[d\mu(P_{L_1}x - \lambda a, *)(P_{X_1}x)/d\mu(P_{L_1}x, P_{X_1}x)] \quad (mod\ \mu). \tag{3}$$

From this Condition (iv) follows.

3.13. Note. For $\mathbf{K_s}$-valued measures Theorem 3.12 is untrue, since there are cases with μ that are not absolutely continuous relative to $\tilde{\mu}$. For example, for a probability measure $\mu = \mu_1 + \mu_2 \neq 0$, $\mu_2(X) = 0$ with μ^1_1 equivalent to the Haar measure m on L_1, $\mu^1_2 = 0$, when for μ_2 all atoms $(a_{j,2})$ are points and for each $x \in L_1$ is accomplished $\sum_{P_{L_1}(a_{j,2})=x} \mu_2(a_{j,2}) = 0$ and atoms $(a_{j,1})$ of μ_1 are such that $P_{L_1}(a_{j,1}) \neq P_{L_1}(a_{l,2})$ for each j,l. We can choose $(a_{j,s})$ such that there exists a compact $S \subset X$ with $\mu_1(S) = 0$, $\mu_2(S) \neq 0$, but $\tilde{\mu}(S) = 0$, since $\mu^1_2 = 0$.

3.14. Definition and notes. A measure $\mu : Bf(X) \to \mathbf{R}$ for a Banach space X over $\mathbf{K}$ is called a quasi-invariant measure if M_μ contains a $\mathbf{K}$-linear manifold J_μ dense in X.

From § 3.7 and Definition 3.11 it follows that $J_\mu \subset M^{\mathbf{K}}_\mu$.

Let $(e_j : j \in \mathbf{N})$ be an orthonormal basis in X, $H = span_{\mathbf{K}}(e_j : j)$. We denote $\Omega := \Omega(\mathbf{R}) := [\mu|\mu$ is a real measure with a finite total variation on $\mathcal{B} = Bf(X)$ and $H \subset J_\mu]$.

A measure $\mu \in \Omega(\mathbf{R})$ that can not be represented as a sum of two singular to each other measures from $\Omega(\mathbf{R})$ is called an extremal measure. $Bf(X)$-measurable function $h : X \to \mathbf{R}$ is called invariant for $\mu \in \Omega(\mathbf{R})$, if $h(a+x) = h(x)$ μ-almost everywhere by x for each $a \in H$.

3.14.1. Lemma. *If ν and μ are measures from Ω and ν^1 is an absolutely continuous component of ν relative to μ, then $\nu^1 \in \Omega$ and $\nu - \nu^1 \in \Omega$ as soon as $\nu \neq \nu^1$.*

Proof. Put $\nu^2 = \nu - \nu^1$, $\nu^2 \perp \mu$. If $a \in H$, then $\nu^1_a + \nu^2_a \sim \nu^1 + \nu^2$ and $\nu^2_a \ll \nu^1 + \nu^2$. Since $\nu^2_a \perp \mu_a \sim \mu$ and $\nu^1 \ll \mu$, then $\nu^2_a \perp \nu^1$ and $\nu^2_a \ll \nu$, that is $\nu^2 \in \Omega$. Then $\nu^1_a \ll \nu^1 + \nu^2$ and $\nu^1_a \perp \nu^2$, consequently, $\nu^1_a \ll \nu^1$, $\nu^1 \in \Omega$.

3.14.2. Corollary. *If μ and ν are two extremal measures, then either $\mu \sim \nu$ or $\mu \perp \nu$.*

Proof. In the contrary case there would be $\mu = \mu^1 + \mu^2$, where $\mu^1 \ll \nu$, $\mu^2 \perp \nu$, but this is impossible, if μ is the extremal measure.

3.14.3. Corollary. *If μ is an extremal measure, then for each $a \in X$ either $\mu \sim \mu_a$ or $\mu \perp \mu_a$.*

Proof. This follows from the fact that μ_a is the extremal measure as well.

3.14.4. Corollary. *If μ is an extremal measure, $\nu \in \Omega$ and $\nu \ll \mu$, then ν is also extremal and $\nu \sim \mu$.*

Proof. In the contrary case $\mu = \mu^1 + \mu^2$, where $\mu^1 \ll \nu$ and $\mu^2 \perp \nu$, $\mu^1, \mu^2 \in \Omega$.

3.14.5. Lemma. *For a measure μ to be extremal it is necessary and sufficient for each invariant relative to μ function $h(x)$ being a constant.*

Proof. Let $h(x)$ be invariant for μ and there exists b such that $\mu(\{x : h(x) < b\}) > 0$ and $\mu(\{x : h(x) \geq b\}) > 0$. Put $\phi_1(x) = 0$ for $h(x) < b$, while $\phi_2(x) = 0$ for $h(x) \geq b$, $\phi_1(x) + \phi_2(x) = 1$, $\mu^j(A) = \int_A \phi_j(x)\mu(dx)$. Evidently, functions $\phi_j(x)$ are also invariant for μ, $j = 1,2$. Therefore, for $a \in H$ we get

$$\int_X f(x)\mu^j_a(dx) = \int_X f(x+a)\mu^j(dx) = \int_X f(x+a)\phi_j(x)\mu(dx)$$
$$= \int_X f(x+a)\phi_j(x+a)\mu(dx) = \int_X f(x)\phi_j(x)\rho_\mu(a,x)\mu(dx)$$
$$= \int_X f(x)\rho_\mu(a,x)\mu^j(dx)$$

so that $\mu^j_a \ll \mu^j$ and hence $\mu^j \in \Omega$, also $\mu^1 \perp \mu$. Since $\mu = \mu^1 + \mu^2$, then μ will not be the extremal measure.

Let now $\mu = \mu^1 + \mu^2$, where $\mu^j \in \Omega$, $\mu^1 \perp \mu^2$. Denote $\phi_j(x) = d\mu^j(x)/d\mu(x)$. For each $a \in H$ there are identities $\phi_1(x)\phi_2(x+a) = \phi_1(x)\phi_2(x) = \phi_1(x+a)\phi_2(x) = 0$ satisfied μ-almost everywhere, since $\mu^1 \perp \mu^2$, $\mu^1_a \perp \mu^2$, $\mu^2_a \perp \mu^1$ and $\phi_1(x) + \phi_2(x) = 1$. Therefore, $0 = \{(\phi_1(x) - \phi_1(x+a)) + (\phi_2(x) - \phi_2(x+a))\}^2 = (\phi_1(x) - \phi_1(x+a))^2 + (\phi_2(x) - \phi_2(x+a))^2$. Thus $\phi_j(x)$ would be invariant different from constants functions for μ.

3.15. Denote by L_n the subspace $span_{\mathbf{K}}\{e_1, \dots, e_n\}$, by X_n the orthogonal in the non-Archimedean sense complement to L_n, $X_n = cl_X\ span_{\mathbf{K}}\{e_j : j > n\}$, by $\mathcal{B}_n$ and $\mathcal{B}^n$ we denote σ-algebras generated by cylindrical subsets in L_n and X_n respectively. As usually $\mu(*, \mathcal{U}|x)$ denotes a conditional measure corresponding to μ relative to a σ-subalgebra $\mathcal{U}$ in $Bf(X)$ on the measure space $(X, Bf(X), \mu)$.

Lemma. *If $h(x)$ is a bounded invariant function for a measure μ, then $h(x) = \int_X h(y)\mu(dy, \mathcal{B}^n|x)$ for all $n \in \mathbf{N}$ and μ-almost everywhere.*

Proof. From § 2.37 the existence of a conditional measure follows. It is sufficient to demonstrate, that there exists a $\mathcal{B}^n$-measurable function $\tilde{h}(x)$ μ-almost everywhere coinciding with $h(x)$. Let $\phi_b(y) = C(b)\exp(-|y|/b)$, where $b > 0$, $C(b) > 0$ is a constant so that $\int_{L_n} \phi_b(y)m_n(dy) = 1$, m_n denotes the Haar measure on L_n with $m_n(B) = 1$ for the unit ball B in L_n. Put $h_b(x) := \int_{L_n} h(x+y)\phi_b(y)m_n(dy)$. From the $Bf(X)$-measurability of $h(x)$ it follows that $h_b(x) \to h(x)$ for $b \to 0$ by the measure μ due to Lemmas 2.8 and 2.30. Therefore, it is possible to choose such sequence b_k that $h_{b_k}(x) \to h(x)$ μ-almost everywhere. Mention that the function $h_b(x+a) = \int_{L_n} h(x+y)\phi_b(y-a)m_n(dy)$ is the continuous function by $a \in L_n$. But due to invariance of h we infer that $h_b(x+a) = h_b(x)$ for μ-almost all x, consequently, $h_b(x+a)$ is constant as the function by a for μ-almost all x by the measure μ. This means that $h_b(x) = h_b(P_n x)$ for μ-almost all $x \in X$, where $P_n : X \to X_n$ is the $\mathbf{K}$-linear projection operator associated with the basis $\{e_j : j\}$. The function $\tilde{h}(x) = \lim_{k\to\infty} h_{b_k}(P_n x)$ is defined for all x, for which this limit exists. It is the desired function.

3.15.1. Corollary. *Let $\mathcal{B}^n_\mu$ be the completion of $\mathcal{B}^n$ by the measure μ, $\mathcal{B}^\infty = \bigcap_n \mathcal{B}^n_\mu \cap \mathcal{B}$. If h is an invariant function for the measure μ, then h is $\mathcal{B}^\infty$-measurable.*

Proof. In view of Lemma 3.15 the invariant function $h(x)$ is $\mathcal{B}^n$-measurable, since such is the bounded invariant function $\arctan(h(x))$.

3.15.2. Corollary. *A function $h(x)$ is invariant for a measure μ if and only if $h(x)$ is $\mathcal{B}^\infty$-measurable.*

Proof. The necessity follows from Corollary 3.15.1. The sufficiency follows from that for a $\mathcal{B}^\infty$-measurable function $h(x)$ for each n there exists a $\mathcal{B}^n$-measurable function $h_n(x)$ so that $\mu(\{x : h(x) = h_n(x)\}) = \mu(X) > 0$ and for each $\mathcal{B}^n$-measurable function $h_n(x)$ for all $y \in L_n$ there is the equality $h_n(x+y) = h_n(x)$.

3.15.3. Corollary. *A probability measure is extremal if and only if μ takes only two values* 0 *and* 1 *on $\mathcal{B}^\infty$.*

Proof. Only in this case all $\mathcal{B}^\infty$-functions are equivalent to constants.

3.16. Theorem. *A measure $\mu \in \Omega(\mathbf{R})$ is an extremal measure if and only if μ is equivalent to $\nu \in \mathsf{R}$, where $\mathsf{R} := [\nu \in \Omega(\mathsf{R})|$ for each n there exists $m > n$ such that $\nu(A \cap B) = \nu(A)\nu(B)$ for each $A \in \mathsf{B}_n$, $B \in \mathsf{B}^m]$.*

Proof. The necessity. Let μ be an extremal measure. Denote by $\mu_n^m(*|y)$ for all $n \leq m$ the conditional measure $\mu(*, \mathcal{B}_n|y)$ on the σ-algebra $calB^m$ and by μ^m the restriction of μ on $\mathcal{B}^m$.

We show that $\mu_n^m(*|y) \sim \mu^m$ for μ-almost all y. It is sufficient to show that $\mu_n^n(*|y) \sim \mu^n$ for μ-almost all y, since $\mu_n^m(*|y)$ and μ^m are restrictions of these measures on $\mathcal{B}^m$.

Let $\lambda^n(A|y)$ be defined for $A \in Bf(X)$, $A \subset X^n$, $y \in L_n$ by the formula:

$$\lambda^n(A|y) = \mu_n^n((P^n)^{-1}A|P_n^{-1}y), \tag{1}$$

where $P_n : X \to L_n$ and $P^n : X \to X_n$ are the projection operator. We denote by λ^n the projection of μ on X_n and λ_n the projection of μ on L_n. It is sufficient to verify that for μ^n-almost all y measures $\lambda^n(*|y)$ and λ^n are equivalent. If consider X as $L_n \times X_n$, then $\mu \sim \lambda_n \times \lambda^n$. Evidently $L_n \in Bf(X)$, since $L_n = \bigcap_{k=1}^\infty L_n^{1/p^k}$, where A^b denotes the b-enlargement of a set A in X, $b > 0$. For each $A \subset L_n$ with $A \in Bf(X)$ there is the equality $\mu(A \times B) = \int_A \lambda_n(dy)\lambda^n(B|y)$ and $\lambda_n \times \lambda^n(A \times B) = \int_A \lambda_n(dy)\lambda^n(B)$, then for proving the relation $\lambda^n(*|y) \sim \mu^n$ for μ_n-almost all y it remains to use Theorem 3.5.1. Thus $\mu_n^m(*|y) \sim \mu^m$ $(mod\ \mu)$.

Let now $\mu_n^\infty(*|y)$ and $\mu^P\infty$ be restrictions of these measures on $\mathcal{B}^\infty$. Then $\mu_n^\infty(*|y) \sim \mu^\infty$ $(mod\ \mu)$. In view of Corollary 3.15.3 the measure μ^∞ on $\mathcal{B}^\infty$ can take only two values 0 and 1. Then the measure $\mu_n^\infty(*|y)$ being equivalent to it coincides with μ^∞, that is $\mu_n^\infty(*|y) = \mu^\infty$ $(mod\ \mu)$.

If $y \in X$ is such that $\mu_n^n(*|y) \sim \mu^n$, then for all $m \geq n$ we get $\mu_n^m(*|y) \sim \mu^m$ and $\mu_n^\infty(*|y) = \mu^\infty$. Denote the set of such y by C_n. For $y \in C_m$ we put $\rho_n^m(x|y) = d\mu_n^m(*|y)(x)/d\mu^m(x)$.

In accordance with Theorem 3.2.4 we infer that

$$\rho_n^{m+1}(x|y) = \int_X \rho_n^m(z|y)\mu(dz, \mathcal{B}^{m+1}|x) \tag{2}$$

for each $y \in C_n$. Equality (2) implies that the sequence of functions $g_k(x) = \rho_n^{N-k}(x,y)$, $k = 1, \ldots, N-n$ forms the martingale, where $N > n$.

In view of Note 2.45 we have that if $A_{(\beta_1,\beta_2)}$ with $\beta_2 > \beta_1 \geq 0$ is the set of those $x \in X$ for which the sequence $\rho_n^m(x,y)$ an infinite number of times intersects the interval (β_1, β_2) for fixed n, x, y, then $\mu(A_{(\beta_1,\beta_2)}) = 0$, since $A_{(\beta_1,\beta_2)} = \bigcap_{k=1}^\infty A_{(\beta_1,\beta_2)}^k$, where $A_{(\beta_1,\beta_2)}^k$ is the set of those $x \in X$ for which the sequence $\rho_n^m(x,y)$ for fixed n, x, y intersects the interval (β_1,β_2) not less than k times, while

$$\mu(A^k_{(\beta_1,\beta_2)}) \le \left[2\int_X \rho_n^m(x|y)dx\right]/[\beta_2+(k-1)(\beta_2-\beta_1)]$$

$$= 2/[\beta_1+k(\beta_2-\beta_1)]. \quad (3)$$

The latter inequality follows from $A^k_{(\beta_1,\beta_2)} = \bigcup_{N=n+1}^{\infty} A^{k,N}_{(\beta_1,\beta_2)}$, where $A^{k,N}_{(\beta_1,\beta_2)}$ is the set of those $x \in X$ for which the sequence $\rho_n^n(x|y),\dots,\rho_n^N(x|y)$ intersects the interval (β_1,β_2) not less than k times. Then $\rho_n^N(x|y)$, $\rho_n^{N-1}(x|y),\dots,\rho_n^n(x|y)$ is the martingale. Apply Note 2.45.

In accordance with Lemma 2.41 we infer that the sequence $\{\rho_n^m(x|y) : m = n, n+1,\dots\}$ is bounded $(mod\ \mu)$. From (2) and the Jensen's inequality we deduce that

$$\int_X \psi(\rho_n^{m+1}(x|y))\mu(dx) \le \int_X \psi(\rho_n^m(x|y))\mu(dx)$$

$$= \int_X \psi(\rho_n^n(x|y))\mu(dx)$$

for each downward convex function ψ. Therefore the sequence $\{\rho_n^m(x|y) : m\}$ is uniformly integrable.

For each $\mathcal{B}^\infty$-measurable function $h(x)$ we have the equalities:

$$\lim_{m\to\infty}\int_X h(x)\rho_n^m(x|y)\mu(dx) = \lim_{m\to\infty}\int_X h(x)\mu_n^m(dx|y)$$

$$= \int_X h(x)\mu_n^\infty(dx|y) = \int_X h(x)\mu(dx)$$

whenever these integrals exist, consequently, $\lim_{m\to\infty}\rho_n^m(x|y) = 1$ $(mod\ \mu)$ for each $y \in C_0$. The function $\rho_n^m(x|y)$ by y is $\mathcal{B}_n$-measurable, as the function by x it is $\mathcal{B}^n$-measurable. Therefore, there exists a $\mathcal{B}^n \times \mathcal{B}_n$-measurable function $\eta_n^m(x|y)$ so that it is equal to $\rho_n^m(x|y)$ by $(mod\ \mu)$. Since ρ_n^m are uniformly integrable, then

$$\lim_{m\to\infty}\int_X |\rho_n^m(x|y)-1|\lambda^n(dx) = \lim_{m\to\infty}\int_X |\rho_n^m(x|y)-1|\mu(dx) = 0. \quad (4)$$

In accordance with Theorem 3.5.1 we can consider that $\rho_n^m(x|y)$ is $\mathcal{B}^m \times \mathcal{B}_n$. Integrating (4) by $\mu(dy)$ we infer that

$$\lim_{m\to\infty}\int_X\int_X |\rho_n^m(x|y)-1|\lambda^n(dx|y) = 0. \quad (5)$$

Consider the measure $\gamma_n := \lambda^n \times \mu_n$. From (5) we get that

$$\lim_{m\to\infty}\int_X\int_X |\rho_n^m(x|y)-1|\gamma_n(dx\times dy) = 0. \quad (6)$$

From the definition we have also $\rho_n^n(x|x) = d\mu(x)/d\gamma_n(x)$. Introduce further the conditional measures $\mu(*,\mathcal{B}^n|y)$. If μ_n and $\mu_n(*,\mathcal{B}^n|y)$ are restrictions of measures μ and

$\mu(*, \mathcal{B}^n|y)$ on the σ-algebra $\mathcal{B}_n$, then $\mu_n(*, \mathcal{B}^n|y) \sim \mu_n$ for μ^n-almost all y as for $\mu_n^m(*|y)$ and $d\mu_n(*, \mathcal{B}^n|y)/d\mu_n(x) = \rho_n^n(x|x)$.

For each $l < n < m$ denote by $\kappa_{l,n}^m$ the restriction of γ_n onto the σ-algebra $\mathcal{B}^l \cap \mathcal{B}_m$, while ν_l^m denotes the restriction of the measure μ onto the same σ-algebra. Put $\pi_{l,n}^m(x) = d\nu_l^m(x)/d\kappa_{l,n}^m(x)$, then $\pi_{l,n}^m(x) = \int_X \rho_n^n(z|z)\gamma_n(dz, \mathcal{B}^l \cap \mathcal{B}_m|x)$ due to Theorem 3.2.4. Applying Theorem 3.2.1 we infer that γ_n-almost everywhere there exists

$$\lim_{m\to\infty} \pi_{l,n}^m(x) = \int_X \rho_n^n(z|z)\gamma_n(dz, \mathcal{B}^l|x) = d\mu^l(x)/d\gamma_n^l(x), \tag{7}$$

where γ_n^l denotes the restriction of the measure γ_n onto the σ-algebra $\mathcal{B}^l$.

For all $l < n < m < k$ define the σ-algebra $\mathcal{B}_{l,n}^{m,k}$ consisting of all sets $A \cap B$ with $A \in \mathcal{B}^l \cap \mathcal{B}_n$, $B \in \mathcal{B}^m \cap \mathcal{B}_k$, we put

$$f_{l,n}^{m,k}(x) = \int_X [\rho_n^m(z|z)]^{-1}\mu(dz, \mathcal{B}_{l,n}^{m,k}|x)\pi_{l,m}^k(x)/[d\mu^l(x)/d\gamma_m^l(x)]. \tag{8}$$

From $(6,7)$ it follows that for all marked l and n

$$\lim_{m\to\infty}\lim_{k\to\infty} f_{l,n}^{m,k}(x) = 1 \quad (mod\ \mu). \tag{9}$$

We rewrite (8) in the form

$$f_{l,n}^{m,k}(x) = [d\gamma_m(\mathcal{B}^l, *)(x)/d\mu(\mathcal{B}^l, *)(x)][d\mu(\mathcal{B}^l \cap \mathcal{B}_k, *)(x)/d\gamma_m(\mathcal{B}^l \cap \mathcal{B}_k, *)(x)] \tag{10}$$

$$\times [d\gamma_n(\mathcal{B}_{l,n}^{m,k}, *)(x)/d\mu(\mathcal{B}_{l,n}^{m,k}, *)(x)],$$

where $\gamma_m(\mathcal{U}, *)$ and $\mu(\mathcal{U}, *)$ denote the restrictions of the measures γ_m and μ onto a σ-algebra $\mathcal{U}$, where $\mathcal{U} \subset \mathcal{B}$. Therefore, $\int_X f_{l,n}^{m,k}(x)\mu(dx) = 1$. If $n_1 < n_2 < \cdots < n_{2N+1}$ is an arbitrary sequence of numbers, then putting $\phi_k(x) = f_{n_{2k-2},n_{2k-1}}^{n_{2k},n_{2k+1}}(x)$ with $n_{2k-2} = 0$ for $k = 1$, $\mathcal{B}^0 = \mathcal{B}$, we get

$$\int_X \prod_{k=1}^N \phi_k(x)\mu(dx) = \int_X \prod_{k=2}^N \phi_k(x)\mu(\mathcal{B}^2, dx) = \cdots$$

$$= \int_X \phi_N(x)\mu(\mathcal{B}^{2N-2}, dx) = 1. \tag{11}$$

Each function ϕ_k is μ-almost everywhere positive. From (9) it follows, that there exists a sequence $\{n_k : k\}$ such that μ-almost everywhere the infinite product $\prod_{k=1}^\infty \phi_k(x) =: g(x)$ converges. Introduce the measure $\zeta(A) := \int_X g(x)\mu(dx)$. From (11) we deduce that $\int_X g(x)\mu(dx) = 1$, hence ζ is the finite measure. Moreover, $\mu \sim \zeta$, since $g(x) > 0$ $(mod\ \mu)$. For a bounded $\mathcal{B}^{n_{2k-1}}$-measurable function ψ and a bounded $\mathcal{B}_{n_{2k-1}}$-measurable function ξ we get

$$\int_X \psi(x)\xi(x)\zeta(dx) = \int_X \psi(x)\xi(x)\prod_{j=1}^\infty \phi_j(x)\mu(dx)$$

$$= \int_X \psi(x)\prod_{j=1}^k \phi_j(x)\xi(x)\prod_{j=k+1}^\infty \phi_j(x)\mu(dx).$$

The function $\xi(x)\prod_{j=k+1}^{\infty}\phi_j(x)$ is $\mathcal{B}^{n_{2k}}$-measurable, while $\xi(x)\prod_{j=1}^{k}\phi_j(x)$ is $\mathcal{B}_{n_{2k+1}}$-measurable. Therefore, the function

$$z_{k+1}(x) := \int_X \xi(z) \prod_{j=k+1}^{\infty} \phi_j(z)\mu(dz, \mathcal{B}_{n_{2k+1}}|x)$$

is $\mathcal{B}^{n_{2k}} \cap \mathcal{B}_{n_{2k+1}}$-measurable and

$$\int_X \psi(x)\xi(x)\zeta(dx) = \int_X \psi(x) \prod_{j=1}^{k} \phi_j(x) z_{k+1}(x)\mu(\mathcal{B}_{n_{2k+1}}|dx). \tag{12}$$

Put $z^{k-1}(x) := \int_X \psi(z)\prod_{j=1}^{k-1}\phi_j(z)\mu(dz, \mathcal{B}^{n_{2k-2}}|x)$. Since $z_{k-1}(x)$ is $\mathcal{B}_{n_{2k-1}}$-measurable, then $z^{k-1}(x)$ is $\mathcal{B}^{n_{2k-2}} \cap \mathcal{B}_{n_{2k-1}}$-measurable. Thus Equation (12) can be rewritten in the form:

$$\int_X \psi(x)\xi(x)\zeta(dx) = \int_X z^{k-1}(x) z_{k+1}(x)[d\gamma_{n_{2k}}(\mathcal{B}^{n_{2k-2}}, *)(x)/d\mu(\mathcal{B}^{n_{2k-2}}, *)(x)]$$

$$\times[d\mu(\mathcal{B}^{n_{2k-2}} \cap \mathcal{B}_{n_{2k+1}}, *)(x)/d\gamma_{n_{2k}}(\mathcal{B}^{n_{2k-2}} \cap \mathcal{B}_{n_{2k+1}}, *)(x)]$$

$$\times[d\gamma_{n_{2k-1}}(\mathcal{B}^{n_{2k},n_{2k+1}}_{n_{2k-2},n_{2k-1}}, *)(x)/d\mu(\mathcal{B}^{n_{2k},n_{2k+1}}_{n_{2k-2},n_{2k-1}}, *)(x)]\mu(\mathcal{B}_{n_{2k+1}}, dx)$$

$$= \int_X z^{k-1}(x) z_{k+1}(x)\gamma_{n_{2k-1}}(dx) = \int_X z^{k-1}(x)\mu(dx) \int_X z_{k+1}(x)\mu(dx)$$

$$= \int_X \psi(x) \prod_{j=1}^{k-1} \phi_j(x)\mu(dx) \int_X \xi(x) \prod_{j=k+1}^{\infty} \phi_j(x)\mu(dx).$$

Particularly for $\psi = 1$ we get

$$\int_X \xi(x)\zeta(dx) = \int_X \xi(x) \prod_{j=k+1}^{\infty} \phi_j(x)\mu(dx),$$

for $\xi = 1$ from (12) we obtain

$$\int_X \psi(x)\zeta(dx) = \int_X \psi(x) \prod_{j=1}^{k-1} \phi_j(x)\mu(dx).$$

This means that

$$\int_X \psi(x)\xi(x)\mu(dx) = \int_X \psi(x)\zeta(dx) \int_X \xi(x)\zeta(dx)/\zeta(X). \tag{13}$$

The measure $w(A) := \zeta(A)/\zeta(X)$ is equivalent with ζ, $w \sim \zeta$. In accordance with (13) for each pair of sets $A \in \mathcal{B}_{n_{2k-1}}$, $B \in \mathcal{B}^{n_{2k}}$ the equalities

$$w(A \cap B) = \int_X \chi_A(x)\chi_B(x)w(dx)$$

$$= \int_X \chi_A(x)w(dx) \int_X \chi_B(x)w(dx) = w(A)w(B)$$

are satisfied, hence $w \in \mathsf{R}$.

The sufficiency. Suppose that $\mu \in \mathsf{R}$. We show that μ is the extremal measure. Let $h(x)$ be a $\mathcal{B}^{\infty}$-measurable bounded function. Then from the definition of the family R it follows that for each $\mathcal{B}_n$-measurable function $\psi(x)$ the equality

$$\int_X \psi(x)h(x)\mu(dx) = \int_X \psi(x)\mu(dx) \int_X h(x)\mu(dx) \tag{14}$$

is satisfied. The latter relation is satisfied for the set of functions $\psi(x)$ closed relative to the bounded convergence. Thus (14) is satisfied for all bounded $\mathcal{B}$-measurable functions. Put in (14) $\psi = h$, then $\int_X h^2(x)\mu(dx) = (\int_X h(x)\mu(dx))^2$, consequently, $\int_X (h(x) - \int_X h(x)\mu(dx))^2\mu(dx) = 0$, hence $h(x) = \int_X h(x)\mu(dx)$ $(mod\ \mu)$. Therefore, each $\mathcal{B}^{\infty}$-measurable function coincides with a constant by the measure μ. Using Lemma 3.14.5 and Corollaries 3.15.1-3 we infer the sufficiency.

3.17. Theorem. *For a Banach space X over $\mathbf{K}$ and each $\mu \in \Omega(\mathbf{R})$ there exists $[\mu^y : y \in Y] \subset \mathsf{R}$ and a measure ν on B^{∞} such that $\mu(A) = \int_X \mu^y(A)\nu(dy)$ for each $A \in Bf(X)$, where $\mathsf{B}^{\infty} := (\bigcap_n \overline{\mathsf{B}}^n \cap Bf(X)$, $\overline{\mathsf{B}}^n$ is a completion of B^n by a measure μ, $\nu = \mu|\mathsf{B}_{\infty}$, $\mu^y(A)$ is $Af(X,\mu)$-measurable (see the notation in* §§ 2.1, 3.14 *and* 3.16).

Proof. We show that the measure $\mu \in Omega$ can be obtained by mixing of extremal measures so that there exists a family of measures $\mu^y \in \mathsf{R}$ on $(X,\mathcal{B})$ for each $y \in Y$ and a measure ν on $(Y,\mathcal{E})$ for which

$$\mu = \int_Y \mu^y \nu(dy), \tag{1}$$

where $\mu^y(B)$ is $\mathcal{E}$-measurable as the function by y for each marked subset $B \in \mathcal{B}$.

Take $(Y,\mathcal{E}) = (X,\mathcal{B}^{\infty})$ and $\nu = \mu|_{\mathcal{B}^{\infty}}$. We put $\mu^y(B) := \mu(B,\mathcal{B}^{\infty}|y)$, where $\mu(*,\mathcal{B}^{\infty}|y)$ is the conditional measure corresponding to μ relative to the σ-algebra $\mathcal{B}^{\infty}$. From § 2.37 for the conditional measure it follows that

$$\mu(A) = \int_X \mu(A,\mathcal{B}^{\infty}|y)\mu(dy) = \int_X \mu^y(A)\nu(dy)$$

for each $A \in \mathcal{B}$.

It remains to show that $\mu^y \in \mathsf{R}$ for ν-almost all y. We first demonstrate that $\mu^y \in \Omega$. Take $a \in J_\mu$ and $A \in \mathcal{B}^{\infty}$ and a cylindrical (measurable) function f on X, then

$$\int_X \int_A f(x+a)\mu^y(dx)\nu(dy) = \int_X f(x+a)\chi_A(x)\mu(dx)$$

$$= \int_X f(x)\chi_A(x)\rho_\mu(a,x)\mu(dx) = \int_X \int_A f(x)\rho_\mu(a,x)\mu^y(dx)\nu(dy).$$

Hence for each such f and ν-almost all y we get:

$$\int_X f(x+a)\mu^y(dx) = \int_X f(x)\rho_\mu(a,x)\mu^y(dx). \tag{2}$$

Take a countable family $\mathcal{C}_f$ of such cylindrical functions f for which (2) is satisfied and the closure of $\mathbf{R}$-linear span of which is dense in the set of all cylindrical real-valued functions on X relative to the point-wise uniformly bounded convergence. Therefore, (2) is satisfied for all cylindrical functions for ν-almost all y, hence

$$d\mu^y_a(x)/d\mu^y(x) = \rho_\mu(a,x) \quad (mod\ \nu) \tag{3}$$

also $\mu_a^y \sim \mu^y$ for each $a \in J_\mu$ and for ν-almost all y.

Denote by m_1 the non-negative Haar measure on $\mathbf{K}$ so that $m_1(B(\mathbf{K},0,1)) = 1$ and let $m_1 \times \nu$ be the measure on the product $(\mathbf{K}, Bf(\mathbf{K}) \times (Y, \mathcal{E}))$, where $Bf(\mathbf{K})$ denotes the Borel σ-algebra on the field $\mathbf{K}$. Then for $m_1 \times \nu$-almost all $(t,y) \in \mathbf{K} \times Y$ there is the relation $\mu_{ta}^y \sim \mu^y$. Therefore, for ν-almost all $y \in Y$ the set $S_y \in Bf(\mathbf{K})$ of those $t \in \mathbf{K}$ for which $\mu_{ta}^y \sim \mu^y$ for a marked y has $m_1(\mathbf{K} \setminus S_y) = 0$. Moreover, S_y is the additive group. Suppose that $h \in \mathbf{K} \setminus S_y$, then $(S_y + h) \cap S_y = \emptyset$, where $S_y + h := \{t + h : t \in S_y\}$, but $m_1(S_y \cap B(\mathbf{K},0,r)) = m_1((S_y + h) \cap B(\mathbf{K},0,r))$ for each $|h| < r < \infty$. Thus $S_y = \mathbf{K}$ for ν-almost all y.

This demonstrates that for ν-almost all $y \in Y$ each $a \in J_\mu$ is the admissible directional vector for the measure μ^y, that is by the definition $ta \in J_{\mu^y}$ for each $t \in \mathbf{K}$. Thus $\mu^y \in \Omega$ for ν-almost all $y \in Y$.

We show next that $\mu^y \in \mathsf{R}$ for ν-almost all y. Denote as usually $\mu^y(*, \mathcal{B}^n|x)$ the conditional measure of μ^y relative to the σ-algebra $\mathcal{B}^n$ for a marked y. Consider a cylindrical function f, a function $g_n(x)$ which is $\mathcal{B}^n$-measurable and bounded, also $A \in \mathcal{B}^\infty$. For each $B \in \mathcal{B}^\infty$ we have

$$\mu(A \cap B) = \int_B \mu(A, \mathcal{B}^\infty|y)\mu(dy) = \int_B \chi_A(y)\mu(dy),$$

consequently, $\mu^y(A) = \chi_A(y)\ (mod\ \nu)$ for each $A \in \mathcal{B}^\infty$. Then

$$\int_A \int_X \int_X f(x)\mu^y(dz, \mathcal{B}^n|x) g_n(x)\mu^y(dx)\nu(dy)$$

$$= \int_A \int_X f(x)g_n(x)\mu^y(dx)\nu(dy) = \int_A f(x)g_n(x)\mu(dx)$$

$$= \int_A g_n(x) \int_X f(z)\mu(z, \mathcal{B}^n|x)\mu(dx)$$

$$= \int_X \int_X \chi_A(x)g_n(x) \int_X f(z)\mu(z, \mathcal{B}^n|x)\mu^y(dx)\nu(dy)$$

$$= \int_A \int_X g_n(x) \int_X f(z)\mu(z, \mathcal{B}^n|x)\mu^y(dx)\nu(dy). \tag{4}$$

From (4) we infer that

$$\int_X f(z)\mu^y(dz, \mathcal{B}^n|x) = \int_X f(z)\mu(dz, \mathcal{B}^n|x) \tag{5}$$

for ν-almost all y and each cylindrical function f. Since

$$\lim_{n\to\infty} \int_X f(z)\mu(dz, \mathcal{B}^n|x) = \int_X f(z)\mu(dz) \quad (mod\ \mu),$$

then for all $f \in \mathcal{C}_l$ for all m for ν-almost all y we get

$$\int_X f(P_m x)\mu^y(dx) = \lim_{n\to\infty} \int_X f(P_m z)\mu^y(dz, \mathcal{B}^n|x) \quad (mod\ \mu^y). \tag{6}$$

Therefore, there exists $E \in \mathcal{B}^\infty$ with $\nu(E) = 1$ so that for each $y \in E$ Equality (6) is satisfied for each cylindrical function f.

Suppose that $h(x)$ is a bounded function invariant relative to the measure μ^y, $y \in E$. For each n the function $h(x)$ is $\mathcal{B}^n(y)$-measurable, where $\mathcal{B}^n(y)$ denotes the completion of $\mathcal{B}^n$ by the measure μ^y, that is the minimal σ-algebra generated by $\mathcal{B}^n$ and by all μ^y-null sets, $\mu^y \ge 0$. For each $m < n$ we deduce that

$$\int_X f(P_m x)h(x)\mu^y(dx) = \int_X h(x) \int_X f(P_m z)\mu^y(dz, \mathcal{B}^n|x)\mu^y(dx).$$

Using (6) we find that

$$\int_X f(P_m x)h(x)\mu^y(dx) = \int_X h(x) \int_X f(P_m z)\mu^y(dz)\mu^y(dx)$$

$$= \int_X h(x)\mu^y(dx) \int_X f(P_m x)\mu^y(dx)$$

or that

$$\int_X f(P_m x)[h(x) - \int_X h(z)\mu^y(dz)]\mu^y(dx) = 0.$$

Taking into account the property of the family $\mathcal{C}_l$ we come to the conclusion that $h(x) = \int_X h(z)\mu^y(dz)$ $(mod\ \mu^y)$. This means that each invariant function $h(x)$ for the measure μ^y is μ^y-almost everywhere constant, consequently, $\mu^y \in \mathsf{R}$ for each $y \in E$.

3.18. Theorem. *If $\mu : Bf(Y) \to \mathbf{R}$ is a σ-finite measure on $Bf(Y)$, Y is a complete separable ultrametrizable $\mathbf{K}$-linear subspace such that $co(S)$ is nowhere dense in Y for each compact $S \subset Y$, where $\mathbf{K}$ is an infinite non-discrete non-Archimedean field with a multiplicative ultranorm $|*|_{\mathbf{K}}$. Then from $J_\mu = Y$ it follows that $\mu = 0$.*

Proof. Since μ is σ-finite, then there are $(Y_j : j \in H) \subset Bf(Y)$ such that $Y = \bigcup_{j \in H} Y_j$ and $0 < \|\mu|Bf(Y_j)\| \le 1$ for each j, where $H \subset \mathbf{N}$, $Y_j \cap Y_l = \emptyset$ for each $j \ne l$. If $card(H) = \aleph_0$, then we define a function $f(x) = 1/[2^j|\mu|(Y_j)]$ for real-valued μ. Then we define a measure $\nu(A) = \int_A f(x)\mu(dx)$, $A \in Bf(Y)$. Therefore, $|\nu|(Y) \le 1$ and $J_\nu = Y$, since $f \in L^1(Y, \mu, \mathbf{R})$. Hence it is sufficient to consider μ with $\|\mu\| \le 1$ and $|\mu|(Y) = 1$. For each $n \in \mathbf{N}$ in view of the Radonian property of Y there exists a compact $X_n \subset Y$ such that $|\mu|(Y \setminus X_n) < 1/n$. In Y there is a countable everywhere dense subset $(x_j : j \in \mathbf{N})$, hence $Y = \bigcup_{j \in \mathbf{N}} B(Y, x_j, r_l)$ for each $r_l > 0$, where $B(Y, x, r_l) = [y \in Y : d(x, y) \le r_l]$, d is an ultrametric in Y, i.e. $d(x, z) \le max(d(x, y), d(y, z))$, $d(x, z) = d(z, x)$, $d(x, x) = 0$, $d(x, y) > 0$ for $x \ne y$ for each $x, y, z \in Y$. Therefore, for each $r_l = 1/l$, $l \in \mathbf{N}$ there exists $k(l) \in \mathbf{N}$ such that $|\mu|(X_{n,l}) > 1 - 2^{-l-n}$, where $X_{n,l} := \bigcup_{j=1}^{k(l)} B(Y, x_j, r_l)$, consequently, $|\mu|(Y \setminus X_n) \le 2^{-n}$ for $X_n := \bigcap_{l=1}^{\infty} X_{n,l}$. The subsets X_n are compact, since X_n are closed in Y and the metric d on X_n is completely bounded and Y is complete (see Theorems 3.1.2 and 4.3.29 [Eng86]). Then $0 < |\mu|(X) \le 1$ for $|\mu|(Y \setminus X) = 0$ for $X := span_{\mathbf{K}}(\bigcup_{n=1}^{\infty} X_n)$.

The sets $\tilde{Y}_n = co(Y_n)$ are nowhere dense in Y for $Y_n = \bigcup_{l=1}^{n} X_l$, consequently, $span_{\mathbf{K}} Y_n$ are nowhere dense in Y. Moreover, $(Y \setminus \bigcup_{n=1}^{\infty} Y_n) \ne \emptyset$ is dense in Y due to the Baire category theorem (see 3.9.3 and 4.3.26 [Eng86]). Therefore, $y + X \subset Y \setminus X$ for $y \in Y \setminus X$ and from $J_\mu = Y$ it follows that $|\mu|(X) = 0$, since $|\mu|(y + X) = 0$ (see §§ 2.38 and 3.14 above). Hence we get the contradiction, consequently, $\mu = 0$.

3.19. Corollary. *If Y is a Banach space or a complete countably-ultranormable infinite-dimensional over $\mathbf{K}$ space, $\mu : Bf(Y) \to \mathbf{R}$ and $J_\mu = Y$, then $\mu = 0$.*

Proof. The space Y is evidently complete and ultrametrizable, since its topology is given by a countable family of ultranorms. Moreover, $co(S)$ is nowhere dense in Y for each compact S in Y, since $co(S) = cl(S_{bc})$ is compact in Y and does not contain in itself any open subset in Y due to (5.7.5)[NB85] (for Y over $\mathbf{R}$ and real-valued measures see Theorem 4 § V.5.3[GV61]).

3.20. Theorem. *Let X be a separable Banach space over a locally compact infinite field $\mathbf{K}$ with a nontrivial normalization such that either $\mathbf{K} \supset \mathbf{Q_p}$ or $char(\mathbf{K}) = p > 0$. Then there are probability measures μ on X with values in $\mathbf{R}$ such that μ are quasi-invariant relative to a dense $\mathbf{K}$-linear subspace J_μ.*

Proof. Let $S(j,n) := p^j B(\mathbf{K},0,1) \setminus p^{j+1}B(\mathbf{K},0,1)$ for $j \in \mathbf{Z}$ and $j \le n$, $S(n,n) := p^n B(\mathbf{K},0,1)$, w be the Haar measure on $\mathbf{K}$ considered as the additive group (see [HR79, Roo78]) with values in $\mathbf{R}$. Then for each $c > 0$ and $n \in \mathbf{N}$ there are measures m on $Bf(\mathbf{K})$ such that $m(dx) = f(x)v(dx)$, $\mid f(x) \mid > 0$ for each $x \in \mathbf{K}$ and $\mid m(p^n B(\mathbf{K},0,1)) - 1 \mid < c$, $m(\mathbf{K}) = 1$, $\mid m \mid (E) \le 1$ for each $E \in Bf(\mathbf{K})$, where $v = w$, $v(B(\mathbf{K},0,1)) = 1$. Moreover, we can choose f such that a density $m_a(dx)/m(dx) =: d(m;a,x)$ be continuous by $(a,x) \in \mathbf{K^2}$ and for each $c' > 0$, x and $\mid a \mid \le p^{-n}$: $\mid d(m;a,x) - 1 \mid < c'$. For this we can define f for $v = w$, for example, to be at the beginning locally constant such that $f(x) = a(j,n)$ for $x \in S(j,n)$, where $a(j,n) = r^{n(j-n)}(1-r^{-n})(1-1/p)p^{-n}$ for $j < n$, $a(n,n) = (1-r^{-2n})p^{-n}$ and $m(E) := \sum\{a(j,n)v(p^{n-j}(E \cap S(j,n))/v(p^{n-j}\ S(j,n)) : j \in \mathbf{Z}, j \le n\}$. Then we can take $g(x) = f(x) + h(x)$ and $y(dx) := g(x)v(dx)$, and a continuous $h(x) : \mathbf{K} \to \mathbf{R}$, with $\sup\{\mid h(x)/f(x) \mid : x \in K\} \le c$" and $0 < c" \le 1/p^n$. More generally it is possible to take $g \in L^1(\mathbf{K}, Bf(\mathbf{K}), v, \mathbf{R})$ such that $g(x) \ge 0$ for v-almost every $x \in \mathbf{K}$ and $\|g\| = 1$ and $\prod_{n=1}^\infty (g_n^{1/2} * g_n^{1/2})(y_n) > 0$ converges for each $y = \{y_n : y_n \in \mathbf{K}, n \in \mathbf{N}\}$ in a proper dense subspace J in $X = c_0$, where $g_n(x) := g(x/a_n)$, $\lim_{n\to\infty} a_n = 0$, $0 \ne a_n \in \mathbf{K}$ for each $n \in \mathbf{N}$, then use the Kakutani theorem 3.3.1, since $H(\mu_n, \nu_n) = (g_n^{1/2} * g_n^{1/2})(y_n)$ for the measure $\mu_n(dx) := g_n(x)v(dx/a_n)$ and its shifted measure $\nu_n(dx) := \mu_n(-y_n + dx)$.

Let $\{m(j;dx)\}$ be a family of measures on $\mathbf{K}$ with the corresponding sequence $\{k(j)\}$ such that $k(j) \le k(j+1)$ for each j and $\lim_{i\to\infty} k(i) = \infty$, where $m(j;dx)$ corresponds to the partition $[S(i,k(j))]$. The Banach space X is isomorphic with $c_0(\omega_0, \mathbf{K})$ [Roo78]. It has the orthonormal basis $\{e_j : j = 1,2,\dots\}$ and the projectors $P_j x = (x(1),\dots,x(j))$ onto $\mathbf{K^j}$, where $x = x(1)e_1 + x(2)e_2 + \cdots$. Then there exists a cylindrical measure μ generated by a consistent family of measures $y(j,B) = b(j,E)$ for $B = P_j^{-1}E$ and $E \in Bf(\mathbf{K}^j)$ [Bou63-69, DF91] where $b(j,dz) = \otimes[m(j;dz(i)) : i = 1,\dots,j], z = (z(1),\dots,z(j))$. Let $L := L(t,t(1),\dots,t(l);l) := \{x : x \in X$ and $\mid x(i) \mid \le p^a, a = -t - t(i)$ for $i = 1,\dots,l$, and $a = -k(j)$ for $j > l\}$, then L is compact in X, since X is Lindelöf and L is sequentially compact [Eng86]. Therefore, for each $c > 0$ there exists L such that $\mid \mu(X \setminus L) \mid < c$, since there is $l \in \mathbf{N}$ with $\mid 1 - \prod[m(j;p^{k(j)}B(K,0,1)) : j > l] \mid < c/2$ (or due to the choice of $a(j,n)$).

In view of the Prohorov theorem cited above (see also § IX.4.2[Bou63-69]) for real measures and due to Lemma 2.3 μ has the countably-additive extension on $Bf(X)$, consequently, also on the complete σ-field $Af(X,\mu)$ and μ is the Radon measure.

Let $z' \in span_{\mathbf{K}}\{e_j : j = 1,2,\dots\}$ and $z" = \{z(j) : z(j) = 0$ for $j \le l$ and $z(j) \in S(n,n), j = 1,2,\dots,n = k(j)\}$, $l \in \mathbf{N}$, $z = z' + z"$.

In accordance with Lemma I.1.4 [Roo78] if $\mathbf{K}$ is a complete relative to its norm non-Archimedean field and $f : B \to \mathbf{K}$ is a mapping such that $f(0) = 0$ and $|[f(x) - f(y)]/[x - y] - 1| \le c$ for each $x,y \in B$, where $0 < c < 1$, $B := B(\mathbf{K},0,r)$ is the ball with center at zero of radius $r > 0$, then $f : B \to B$ is the isometry.

By Corollary 2.4 [Roo78] if (X,d) is a complete ultrametric space and if every decreasing sequence of values of the ultrametric d converges to zero, then (X,d) is spherically complete.

In view of the Kakutani theorem cited above (see also [DF91]) and also two statements cited just above there are measures $m(j;dz(j)$ such that $\rho_\mu(z,x) = \prod\{d(j;z(j),x(j)) : j = 1,2,\dots\} = \mu_z(dx)/\mu(dx) \in L^1(X,\mu,\mathbf{R})$ for each such z and $x \in X$, where $d(j;*,*) = d(m(j;*),*,*)$ and $\mu_z(X) = \mu(X) = 1$.

When $char(\mathbf{K}) = 0$, then in this proof it is also possible to consider $S(j,n) := \pi^j B(\mathbf{K},0,1) \setminus \pi^{j+1} B(\mathbf{K},0,1)$, where $\pi \in \mathbf{K}$, $p^{-1} \le |\pi| < 1$, π is the generator of the normalization group of $\mathbf{K}$.

3.20.1. Theorem. *Let X be a Banach space of separable type over a locally compact non-Archimedean field $\mathbf{K}$ and J be a dense proper $\mathbf{K}$-linear subspace in X such that the embedding operator $T : J \hookrightarrow X$ is compact and nondegenerate, $ker(T) = \{0\}$. Then a set $\mathcal{M}(X,J)$ of probability measures μ on $Bf(X)$ quasi-invariant relative to J is of cardinality 2^{c}. If J', $J' \subset J$, is also a dense $\mathbf{K}$-linear subspace in X, then $\mathcal{M}(X,J') \supset \mathcal{M}(X,J)$.*

Proof. Since X is of separable type over $\mathbf{K}$, then we can choose for a given compact operator T an orthonormal base in X in which T is diagonal and X is isomorphic with c_0 over $\mathbf{K}$ such that in its standard base $\{e_j : j \in \mathbf{N}\}$ the operator T has the form $Te_j = a_j e_j$, $0 \ne a_j \in \mathbf{K}$ for each $j \in \mathbf{N}$, $\lim_{j\to\infty} a_j = 0$ (see Appendix). As in Theorem 3.20 take $g_n \in L^1(\mathbf{K}, Bf(\mathbf{K}), v(dx/a_n), \mathbf{R})$, $g_n(x) > 0$ for v-a.e. $x \in \mathbf{K}$ and $\|g_n\| = 1$ for each n, for which converges $\prod_{n=1}^{\infty}(g_n^{1/2} * g_n^{1/2})(y_n)$, where the convolution is taken relative to the measure $v(dx/a_n)$, for each $y \in J$ and such that $\prod_{n=1}^{m} g_n(x_n) v(dx_n/a_n) =: \nu_{L_n}(dx^n)$ satisfies conditions of Lemma 2.3, where $x^n := (x_1,\dots,x_n)$, $x_1,\dots,x_n \in \mathbf{K}$ for each $n \in \mathbf{N}$. Since g_n are nonnegative, then it is sufficient to satisfy conditions of Lemma 2.3 for $r = b > 0$. The family of such sequences of functions $\{g_n : n \in \mathbf{N}\}$ has the cardinality 2^{c}, since in L^1 the subspace of step functions is dense and $card(Bf(X)) = \aleph_0^{\aleph_0} = \mathsf{c}$, $card([0,\infty)^{\mathsf{c}}) = 2^{\mathsf{c}}$, $2^{\mathsf{c}\times\aleph_0} = 2^{\mathsf{c}}$, where $\aleph_0 := card(\mathbf{N})$, $\mathsf{c} := card(\mathbf{R})$. The latter statement of this theorem is evident, since the family of all $\{g_n : n\}$ satisfying conditions above for J also satisfies such conditions for J'.

3.21. Note. For a given $m = w$ new suitable measures may be constructed, if to use images of measures $m^g(E) = m(g^{-1}(E))$ such that for a diffeomorphism $g \in Diff^1(\mathbf{K})$ (see § A.3) we have $m^{g^{-1}}(dx)/m(dx) = |(g'(g^{-1}(x))|_{\mathbf{K}}$, where $|*|_{\mathbf{K}} = mod_{\mathbf{K}}(*)$ is the modular function of the field $\mathbf{K}$ associated with the Haar measure on $\mathbf{K}$, at the same time $|*|_{\mathbf{K}}$ is the multiplicative norm in $\mathbf{K}$ consistent with its uniformity [Wei73]. Indeed, for $\mathbf{K}$ and $X = \mathbf{K}^{\mathbf{j}}$ with $j \in \mathbf{N}$ and the Haar measure $v = w$ or $v = w'$ on X, $v_X := v$ with values either in $\mathbf{R}$ or in $\mathbf{K_s}$ for $s \ne p$ and for a function $f \in L^1(X,v,\mathbf{R})$ or $f \in L(X,v,\mathbf{K_s})$, respectively, due to Lemma 4 and Theorem 4 § I.2[Wei73], Theorems 9.2 and A.7[Sch84] we have: $\int_{g(A)} f(x)v(dx) = \int_A f(g(y))|g'(y)|_{\mathbf{K}} v(dy)$, where $mod_{\mathbf{K}}(\lambda)v(dx) := v(\lambda dx)$, $\lambda \in \mathbf{K}$. For a construction of new measures images of v may be used, for example, for analytic h on $\mathbf{K}$ (see 43.1 [Sch84]) with $g = Ph$, where P is the anti-differentiation such that $g' = h$. Defining it at first locally we can then extend it on all $\mathbf{K}$.

Henceforward, quasi-invariant measure μ on $Bf(c_0(\omega_0,\mathbf{K}))$ constructed with the help of projective limits or sequences of weak distributions of probability measures $(\mu_{H(n)} : n)$ are considered, for example, as in Theorem 3.20 such that

(i) $\mu_{H(n)}(dx) = f_{H(n)}(x)\nu_{H(n)}(dx)$, $dim_{\mathbf{K}}H(n) = m(n) < \aleph_0$ for each $n \in \mathbf{N}$, where $f_{H(n)} \in L^1(H(n), \nu_{H(n)}, \mathbf{R})$, $H(n) \subset H(n+1) \subset \cdots$, $cl(\bigcup_n H(n) = c_0(\omega_0, \mathbf{K})$, if it is not specified in another manner.

Proposition 11 § VII.1.9 [Bou63-69] states, that if G is a locally compact group, G_1 is its closed normal subgroup, $G_2 = G/G_1$ is the quotient group, $\pi : G \to G_2$ is the quotient mapping, $u : G \to G$ is a topological group automorphism of G such that $u(G_1) = G_1$, u_2 is an automorphism of G_2 obtained by the factorization, then $mod_G(u) = mod_{G_1}(u_1)mod_{G_2}(u_2)$.

For real-valued probability quasi-invariant measures for a sequence of weak distributions $(\mu_{H(n)})$ generated by μ in view of the proposition cited just above, Lemma 2.3 and Definition 3.14 this condition is satisfied, since $cl(J_\mu) = c_0(\omega_0, \mathbf{K})$.

As will be seen below such measures μ are quasi-invariant relative to families of the cardinality $\mathsf{c} = card(\mathbf{R})$ of linear and non-linear transformations $U : X \to X$. Moreover, for each V open in X we have $\mu(V) > 0$, when $f_{H(n)}(x) > 0$ for each $n \in \mathbf{N}$ and $x \in H(n)$.

Let μ be a probability quasi-invariant measure satisfying (i) and $(e_j : j)$ be an orthonormal basis in M_μ, $H(n) := span_{\mathbf{K}}(e_1, \dots, e_n)$, we denote by

$$\hat{\rho}_\mu(a,x) = \hat{\rho}(a,x) = \lim_{n\to\infty} \rho^n(P_n a, P_n x),$$

$$\rho^n(P_n a, P_n x) := f_{H(n)}(P_n(x-a))/f_{H(n)}(P_n x)$$

for each a and x for which this limit exists and $\hat{\rho}(a,x) = 0$ in the contrary case, where $P_n : X \to H(n)$ are chosen consistent projectors. Let $\rho(a,x) = \hat{\rho}(a,x)$, if $\int \hat{\rho}(a,x)\mu(dx) = 1$, $\rho(a,x)$ is not defined when $\hat{\rho}(a,x) \neq 1$. If for some another basis $(\tilde{e}_j : j)$ and $\tilde{\rho}$ is accomplished

(ii) $\mu(S) = 1$, $S := \bigcap_{a\in M_\mu}[x : \rho(a,x) = \tilde{\rho}(a,x)])$, then $\rho(a,x)$ is called regularly dependent from a basis.

3.22. Lemma. *Let μ be a probability measure, $\mu : Bf(X) \to \mathbf{R}$, X be a Banach space over $\mathbf{K}$, suppose that for each basis $(\tilde{e}_j : j)$ in M_μ a quasi-invariance factor $\tilde{\rho}$ satisfies the following conditions:*

(1) *if $\tilde{\rho}(a_j, x)$, $j = 1, \dots, N$, are defined for a given $x \in X$ and for each $\lambda_j \in \mathbf{K}$ then a function $\tilde{\rho}(\sum_{j=1}^n \lambda_j a_j, x)$ is continuous by λ_j, $j = 1, \dots, N$;*

(2) *there exists an increasing sequence of subspaces $H(n) \subset M_\mu$, $cl(\bigcup_n H(n)) = X$, with projectors $P_n : X \to H(n)$, $B \in Bf(X)$, $\mu(B) = 0$ such that $\lim_{n\to\infty} \tilde{\rho}(P_n a, x) = \tilde{\rho}(a,x)$ for each $a \in M_\mu$ and $x \notin B$ for which is defined $\rho(a,x)$. Then $\rho(a,x)$ depends regularly from the basis.*

Proof. There exists a subset S dense in each $H(n)$, hence $\mu(B') = 0$ for $B' = \bigcup_{a\in S}[x : \rho(a,x) \neq \tilde{\rho}(a,x)]$. From (1) it follows that $\tilde{\rho}(a,x) = \rho(a,x)$ on each $H(n)$ for $x \notin B'$. From $span_{\mathbf{K}}S \supset H(n)$ and (2) it follows that $\rho(a,x) = \tilde{\rho}(a,x)$ for each $a \in M_\mu$ and $x \in X \setminus (B' \cup B)$, consequently, Condition 3.21(ii) is satisfied.

3.23. Lemma. *If a probability quasi-invariant measure $\mu : Bf(X) \to \mathbf{R}$ satisfies Condition 3.21(i), then there exists a compact operator $T : X \to X$ such that $M_\mu \subset (TX)^\sim$, where X is the Banach space over $\mathbf{K}$.*

Proof. This follows from Theorem 3.9 (see also Chapter II).

3.24. Let X be a Banach space over $\mathbf{K}$, $|*|_{\mathbf{K}} = mod_{\mathbf{K}}(*)$, $U : X \to X$ be an invertible linear operator, $\mu : Bf(X) \to \mathbf{R}$ be a probability quasi-invariant measure.

The uniform convergence of a (transfinite) sequence of functions on $Af(V,\nu)$-compact subsets of a topological space V is called the Egorov condition, where ν is a measure on V.

Theorem. *Let pairs* $(x-Ux,x)$ *and* $(x-U^{-1},x)$ *be in* $dom(\tilde{\rho}(a,x))$*, where* $dom(f)$ *denotes a domain of a function* f*,* $\tilde{\rho}(x-Ux,x)>0$*,* $\tilde{\rho}(x-U^{-1}x,x)>0$ $(mod\ \mu)$*. Then* $\nu\sim\mu$ *and*

$$(i)\ \nu(dx)/\mu(dx)=|det(U)|_{\mathbf{K}}\tilde{\rho}(x-U^{-1}x,x),$$

if ρ *depends regularly from the base, then* $\tilde{\rho}$ *may be substituted by* ρ *in formula* (i)*, where* $\nu(A):=\mu(U^{-1}A)$ *for each* $A\in Bf(X)$*.*

Proof. In view of Theorem 3.9 (and Lemma 3.23) there exists a compact operator $T:X\to X$ such that $M_\mu\subset(TX)^{\sim}$, consequently, $(U-I)$ is a compact operator, where I is the identity operator. From the invertibility of U it follows that $(U^{-1}-I)$ is also compact, moreover, there exists $det(U)\in\mathbf{K}$. Let g be a continuous bounded function, $g:\tilde{H}(n)\to\mathbf{R}$, whence

$$\int_X\phi(x)\nu(dx)=\int_{\tilde{H}(n)}g(x)[f_{H(n)}(U^{-1}x)/f_{\tilde{H}(n)}(x)]|det(U_n)|_{\mathbf{K}}\mu_{\tilde{H}(n)}(dx),$$

for $\phi(x)=g(\tilde{P}_nx)$, where subspaces exist such that $\tilde{H}(n)\subset X$, $(U^{-1}-I)\tilde{H}(n)\subset\tilde{H}(n)$, $cl(\bigcup_n\tilde{H}(n))=X$, $U_n:=\hat{r}_n(U)$, $r_n=\tilde{P}_n:X\to\tilde{H}(n)$ (see §§ 3.8 and 3.21), $\tilde{H}(n)\subset\tilde{H}(n+1)\subset\cdots$ due to compactness of $(U-I)$. In view of the Fatou theorem $J_m\geq J_{m,\rho}$, where $J_m:=\int_Xg(\tilde{P}_mx)\nu(dx)$ and $J_{m,\rho}:=\int_Xg(\tilde{P}_mx)\tilde{\rho}(x-U^{-1}x,x)|det(U)|_{\mathbf{K}}\mu(dx)$. Indeed, there exists n_0 such that $|u(i,j)-\delta_{i,j}|\leq1/p$ for each i and $j>n_0$, consequently, $|det(U_n)|_{\mathbf{K}}=|det(U)|_{\mathbf{K}}$ for each $n>n_0$, where $\delta_{i,j}$ is the Kronecker delta-symbol.

This implies that for each non-negative $Bf(X)$-measurable function $\phi(x)$ the inequality

$$\int_X\phi(x)\nu(dx)\geq\int_X\phi(x)\tilde{\rho}(x-U^{-1}x,x)|det(U)|_{\mathbf{K}}\mu(dx)$$

is satisfied, consequently, $\mu\ll\nu$.

Then due to the Fubini Theorem there exists

$$\lim_{n\to\infty}[\mu_{\tilde{H}(n)}(d\tilde{P}_nx)/\nu_{\tilde{H}(n)}(d\tilde{P}_nx)]=\mu(dx)/\nu(dx)\quad(mod\ \nu).$$

Therefore,

$$d\mu(x)/d\nu(x)=1/[\tilde{\rho}_\mu(x-U^{-1}x,x)|det(U)|_{\mathbf{K}}]\quad(mod\ \mu).\tag{1}$$

This means that the density of the absolute continuous component of ν relative to μ is:

$$d\nu(x)/d\mu(x)=\tilde{\rho}_\mu(x-U^{-1}x,x)|det(U)|_{\mathbf{K}}.\tag{2}$$

Consider the measure λ obtained from μ by the mapping U^{-1}, then $\mu\ll\lambda$ as soon as $\tilde{\rho}_\mu(x-Ux,x)>0$ $(mod\ \mu)$. Under the mapping U the measure μ transforms into ν, while λ into μ. Thus $\nu\ll\mu$ as well and inevitably $\mu\sim\nu$.

3.25. Let X be a Banach space over $\mathbf{K}$, $|*|_{\mathbf{K}}=mod_{\mathbf{K}}(*)$ with a probability quasi-invariant measure $\mu:Bf(X)\to\mathbf{R}$, also let U fulfils the following conditions:

(i) $U(x)$ and $U^{-1}(x)\in C^1(X,X)$ (see § A.3);

(ii) $(U'(x)-I)$ is compact for each $x\in X$;

$$(iii)\ (x - U^{-1}(x)) \text{ and } (x - U(x)) \in J_\mu \text{ for } \mu - \text{a.e. } x \in X;$$

$$(iv) \text{ for } \mu\text{-a.e. } x \text{ pairs } (x - U(x); x) \text{ and } (x - U^{-1}(x); x)$$

are contained in a domain of $\rho(z,x)$ such that $\rho(x - U^{-1}(x), x) \neq 0$, $\rho(x - U(x), x) \neq 0$ $(mod\ \mu)$;

$$(v)\ \mu(S') = 1,$$

where $S' := ([x :\ \rho(z,x)$ is defined and continuous by $z \in L])$ for each finite-dimensional $L \subset J_\mu$;

$$(vi) \text{ there exists } S \text{ with } \mu(S) = 0 \text{ and for each}$$

$x \in X \setminus S$ and for each z for which there exists $\rho(z,x)$ satisfying the following condition: $\lim_{n\to\infty} \rho(P_n z, x) = \rho(z,x)$ and the convergence is uniform for each finite-dimensional $L \subset J_\mu$ by z in $L \cap [x \in J_\mu : |x| \leq c]$, where $c > 0$, $P_n : X \to H(n)$ are projectors onto finite-dimensional subspaces $H(n)$ over $\mathbf{K}$ such that $H(n) \subset H(n+1)$ for each $n \in \mathbf{N}$ and $cl \cup \{H(n) : n\} = X$;

$$(vii) \text{ there exists } n \text{ for which for all } j > n \text{ and } x \in X \text{ mappings}$$

$V(j,x) := x + P_j(U^{-1}(x) - x)$ and $U(j,x) := x + P_j(U(x) - x)$ are invertible and $\lim_j |\det U'(j,x)| = |\det U'(x)|$, $\lim_j |\det V'(j,x)| = 1/|\det U'(x)|$.

Theorem. *The measure* $\nu(A) := \mu(U^{-1}(A))$ *is equivalent to* μ *and*

$$(i)\ \nu(dx)/\mu(dx) = |\det U'(U^{-1}(x))|_{\mathbf{K}}\, \rho(x - U^{-1}(x), x).$$

Proof. I. Let at first U be linear. In general, for a linear operator U with compact $B = U - I$ there is the following decomposition $U = SCDE$, where C^t and E are upper triangular infinite matrices, $D = diag\{d(j) : j \in \mathbf{N}\}$, operators $C - I$, $D - I$ and $E - I$ are compact in the corresponding orthonormal basis $\{e_j : j\}$ in X, S transposes a finite number of vectors in orthonormal basis (see § A.2). Moreover, there are $det(C) = det(E) = 1$, $det(U) = det(D) \neq 0$.

II. Let V_n be a diagonal (or upper or lower triangular) operator on X such that $(V_n - I)(X) = P_n L$, where $dim_{\mathbf{K}} L = k < \aleph_0$, $\lim_{n\to\infty} \|V_n - U_1\| = 0$, U_1 is a diagonal (or lower or upper triangular) operator, there exists n_0 such that $\|e_j - P_n e_j\| \leq 1/p$ for (e_j) in L, consequently, $\|\sum_j \lambda_j P_n e_j\| = \max_j |\lambda_j|$ for $\lambda_j \in \mathbf{K}$ and $dim_{\mathbf{K}} P_n L = k$ for $n > n_0$, in addition, $\lim_{n\to\infty} \sup_{x\in L, \|x\|\leq 1} \|x - P_n x\| = 0$. Then $\lim_n \tilde{P}_n^{-1}(x - V_n^{-1} x) = x - U_1^{-1} x$, where $\tilde{P}_n x = P_n x \in P_n L$ for each $x \in L$. Due to Conditions (vi, vii) we get $\lim_n \rho(x - V_n^{-1} x, x) = \rho(x - U_1^{-1} x, x)$ $(mod\ \mu)$.

From the Fatou theorem and § 3.21 it follows that, $J_1 \geq J_{1,\rho}$, where $J_1 = \int_X f(U_1 x)\mu(dx)$, $J_{1,\rho} := \int_X f(x)\rho(x - U_1^{-1}x, x)|detU_1|_{\mathbf{K}}\mu(dx)$ for continuous bounded function $f : X \to [0,\infty)$. Analogously we proceed for the operator U_1^{-1} instead of U_1. Using instead of f the function $\Phi(U_1^{-1}x) := f(x)\rho_\mu(x - U_1^{-1}x, x)$ and Properties 3.7 we get that $\rho_\mu(U_1 x - x, U_1 x)\rho_\mu(x - U_1 x, x) = 1$ $(mod\ \mu)$. Therefore, for $U = U_1 U_2$ with diagonal U_1 and upper triangular U_2 and lower triangular U_3 operators with finite-dimensional over $\mathbf{K}$ subspaces $(U_j - I)X$, $j = 1,2,3$, the following equation is accomplished

$$\int_X f(Ux)\mu(dx) = \int_X f(x)\rho_\mu(x - U^{-1}x, x)|detU|_{\mathbf{K}}\mu(dx).$$

If $(S^{-1}U-I)X=L$, then from the decomposition given in (I) $U=SU_2U_1U_3$, we have $(U_j-I)X=L$, $j=1,2,3$ due to formulas from § A.1 (see Appendix), since corresponding non-major minors are equal to zero.

(III). If U is an arbitrary linear operator satisfying the conditions of this theorem, then from (iv-vi) and (I, II) for each continuous bounded function $f:X\to[0,\infty)$ we have $J\geq J\rho$, where $J:=\int_X f(U(x))\mu(dx)$ and $J_\rho:=\int_X f(x)\rho_\mu(x-U^{-1}(x),x)|detU|_{\mathbf{K}}\mu(dx)$. Analogously for U^{-1}, moreover, $\rho(x-U^{-1}(x),x)|detU|_{\mathbf{K}}=:h(x)\in L^1(X,\mu,\mathbf{R})$, $h(x)>0$ $(mod$ $\mu)$, since there exists $detU$.

(IV). Now let U be linear and $(U-I)(X)=L$, $dim_{\mathbf{K}}L=k<\aleph_0$, $L\subset J_\mu$. Suppose U is polygonal, which means that there exists a partition $X=\cup\{Y(i):i=1,\dots,l\}$, $U(x)=a(i)+V(i)x$ for $x\in Y(i)$, where $Y(i)$ are closed subsets, $IntY(i)\cap IntY(j)=\emptyset$ for each $i\neq j$, $a(i)\in X$ and $V(i)$ are linear operators. Then U^{-1} is also polygonal, $U'(x)=V(j)$ for $x\in Y(j)$ and $\int_X f(a(i)+V(i)x)\mu(dx)=\int f(a(i)+x)\rho_\mu(x-V^{-1}(i)x,x)\times|det(V(i))|_{\mathbf{K}}\mu(dx)$ for each real Borel measurable function f and each i. From $a(j)\in M_\mu$ and § 3.7 we get $\int_X f(a(j)+V(j)x)\mu(dx)=\int_X f(x)\rho(x-V(j)^{-1}(x-a(j)),x)|detV(j)|_{\mathbf{K}}\mu(dx)$. Let $H_{k,j}:=[x\in X:V(k)^{-1}x=V(j)^{-1}x]$, assume without loss of generality that $V(k)\neq V(j)$ or $a(k)\neq a(j)$ for each $k\neq j$, since $Y(k)\neq Y(j)$ (otherwise they may be united). Therefore, $H_{k,j}\neq X$. If $\mu(H_{k,j})>0$, then from $X\ominus H_{k,j}\supset\mathbf{K}$ it follows that $M_\mu\subset H_{k,j}$, but $cl(H_{k,j})=H_{k,j}$ and $cl(M_\mu)=X$. This contradiction means that $\mu(A)=0$, where $A=[x:V(k)^{-1}(x-a(k))=V(j)^{-1}(x-a(j))]$. Then $\int_X f(U(x))\mu(dx)=\int_X f(x)\rho(x-U^{-1}(x),x)\ |detU'(x)|^{-1}_{\mathbf{K}}\mu(dx)$.

(V). The field $\mathbf{K}$ is spherically complete. In view of the Hahn-Banach theorem for the Banach space X over $\mathbf{K}$ [NB85, Roo78] there are linear continuous functionals $\tilde{e}_j$ such that there exists orthonormal basis $(e_j:j)$ in X with $\tilde{e}_i(e_j)=\delta_{i,j}$. Let $s(i,j;x):=\tilde{e}_i(U(j,x)-x)$ and $s(i;x)=\tilde{e}_i(U(x)-x)=\lim_j s(i,j;x)$ (lim is taken in $C(X,\mathbf{K})$), consequently, $\det U'(j,x)=\det(ds(i,j;x)e(k):i,k=1,\dots,j)$. Then for the construction of the sequence $\{U(j,*):j\}$ it is sufficient to construct a sequence of polygonal functions $\{a(i,j;x)\}$, that is $a(i,j;x)=l_k(i,j)(x)+a_k$ for $x\in Y(k)$, where $l_k(i,j)$ are linear functionals, $a_k\in\mathbf{K}$, $Y(k)$ are closed in X, $Int(Y(j))\cap Int(Y(k))=\emptyset$ for each $k\neq j$, $\bigcup_{k=1}^m Y(k)=X$, $m<\aleph_0$. For each $c>0$ there exists $V_c\subset X$ with $\mu(X\setminus V_c)<c$, the functions $s(i,j;x)$ and $(\bar{\Phi}^1 s(i,j;*))(x,e(k),t)$ are equiuniformly continuous (by $x\in V_c$ and by $i,j,k\in\mathbf{N}$) on V_c. Choosing $c=c(n)=1/n$ and using δ-nets in V_c we get a sequence of polygonal mappings $(W_n:n)$ converging by its matrix elements in $L^1(X,\mu,\mathbf{R})$, from Condition (i) follows that it may be chosen equicontinuous for matrix elements $s(i,j;x)$, $ds(i,j;x)$ and $s(i,P_jx)$ by i,j (the same is true for U^{-1}).

Indeed, for V_c there is $\delta>0$ such that $|s(i,j;x')-s(i,j;x)|<c$ and

$$|(\bar{\Phi}^1 s(i,j;*))(x,e(k),t)-(\bar{\Phi}^1 s(i,j;*))(x',e(k),t')|<c$$

(see also Theorem 2.11.I [Lud99t]) for each $x,x'\in V_c$ with $|x-x'|<\delta$, $|t-t'|\leq 1$ and $i,j,k\in\mathbf{N}$. Then we can choose in V_c a finite δ-net $x_1,\dots,x_r$ and define $l_q(i,j;x)=s(i,j;x_q)+(ds(i,j;x_q))(x-x_q)$ and apply the non-Archimedean variant of the Taylor theorem (see §29.4[Sch84] and Theorem A.5 in the Appendix).

Then calculating integrals as above for W_n with functions f, using the Fatou theorem we get the inequalities analogous to written in (III) for J and J_ρ of the general form. From

$\nu(dx)/\mu(dx) > 0$ $(mod\ \mu)$ we get the statement of this theorem, since

$$\int_X f(A(j,x))\mu(dx) = \int_X f(x)\rho_\mu(P_j(x-A(x)),x)|det(A(j,x)')|_{\mathbf{K}}^{-1}\mu(dx),$$

where $A = U$, $A(j,x) = U(j,x)$ or $A = U^{-1}$, $A(j,x) = V(j,x)$.

3.26. Note. For a linear invertible operator U Condition 3.25(i) is satisfied automatically, (ii) follows from (iii), hence from (iii) also follows (vii) (see the proof of Theorem 3.24).

3.27. Examples. Let X be a Banach space over the field $\mathbf{K}$ with the normalization group $\Gamma_{\mathbf{K}} = \Gamma_{\mathbf{Q_p}}$. We consider a diagonal compact operator $T = diag(t_j : j \in \mathbf{N})$ in a fixed orthonormal basis (e_j) in X such that $kerT := T^{-1}0 = \{0\}$. Let $\nu_j(dx_j) = C(\xi_j)\exp(-|(x_j - x_j^0)/\xi_j|_p^q)v(dx_j)$ for the Haar measure $v : Bf(\mathbf{K}) \to [0,\infty)$. We choose constant functions $C(\xi_j)$ and $C'(\xi_j)$ such that ν_j be a probability measure, where $x^0 = (x_j^0 : j) \in X$, $x = (x_j : j) \in X$, $x_j \in \mathbf{K}$. Particularly, for $q = 2$: $\int_{\mathbf{K}}[(\chi_e(x_j - x_j^0) - \chi_e(-x_j + x_j^0))/(2i)]\nu_j(dx_j) = 0$, since ν_j is symmetric and $[\chi_e(x) - \chi_e(-x)]$ is the odd function, where χ_e is the same character as in § 2.6. Then $\int_{\mathbf{K}} |x_j - x_j^0|^2\nu_j(dx_j) = |\xi_j|^2$ due to the formula of changing variables in § 3.21 and differentiation by the real parameter b of the following integral $I(b) = \int_{\mathbf{K}} \exp(-|x|_p^2 b^2)v(dx)$, then we take $1/|\xi| = b \in \Gamma_{\mathbf{K}}$. Therefore, x_j^0 and $|\xi_j|^2$ have in some respect a meaning looking like the mean value and the variance.

With the help of products $\bigotimes_{j=1}^{\infty}\nu_j(dx_j)$ as in § 3.20 we can construct a probability quasi-invariant measure μ^T on X with values in $[0,1]$ or $\mathbf{C_s}$, since $cl(TX)$ is compact in X and $span_{\mathbf{K}}(e_j : j) =: H \subset J_\mu$. From $\bigcap_{\lambda \in B(\mathbf{K},0,1)\setminus 0} cl(\lambda TX) = \{0\}$ we may infer that for each $c > 0$ there exists a compact $V_c(\lambda) \subset X$ such that $\mu(X \setminus V_c(\lambda)) < c$ and $\bigcap_{\lambda \neq 0} V_c(\lambda) = \{0\}$, consequently, $\lim_{|\lambda|\to 0}\int_X f(x)\mu^{\lambda T}(dx) = f(0) = \delta_0(f)$, hence $\mu^{\lambda T}$ is weakly converging to δ_0 whilst $|\lambda| \to 0$ for the space of bounded continuous functions $f : X \to \mathbf{R}$.

From Theorem 3.4 we conclude that from $\sum_{j=1}^{\infty}|y_j/\xi_j|_p^q < \infty$ it follows $y \in J_{\mu^T}$. Then for a linear transformation $U : X \to X$ from $\sum_j |\tilde{e}_j(x - U(x))/\xi_j|_p^2 < \infty$ it follows that $x - U(x) \in J_\mu$ and a pair $(x - U(x),x) \in dom(\rho(a,z))$. Moreover, for ρ corresponding to μ^T conditions (v) and (vi) in § 3.25 are satisfied. Therefore, for such y and $S \in Af(X,\mu)$ a quantity $|\mu(ty + S) - \mu(S)|$ is of order of smallness $|t|^q$ whilst $t \to 0$, hence they are pseudo-differentiable of order b for $0 < Re(b) < 2$ (see also § 4 below).

Analogs of real-valued Wiener measures on spaces of functions from a non-Archimedean Banach space into $\mathbf{C}$ are given in [BV97, Sat94], that are quasi-invariant under some suitable choice. Another examples of measures with particular properties are given in § 1.6.

3.28. Theorem. *Let A be a complete normed algebra over a locally compact infinite field $\mathbf{K}$ with a non-trivial non-Archimedean multiplicative norm. If a nontrivial real-valued measure μ on $Bf(A)$ is quasi-invariant relative to a dense subalgebra A' (relative to linear shifts and left (or right) multiplication), then A is finite dimensional over $\mathbf{K}$.*

Proof. If A is without a unity, then we can consider the algebra A_1 with unity such that $A \subset A_1$ and A_1 is generated by A and $\mathbf{K}1$. Therefore, μ can be extended to a measure μ' on A_1 quasi-invariant relative to A'_1, where A'_1 is generated by A' and $\mathbf{K}1$, $\mu' = \mu \times m$, where m is the nontrivial Haar measure on $\mathbf{K}$. Therefore, without loss of generality consider A with the unit element. In view of Lemma 3.23 there exists a compact operator T on A such that $M_\mu \subset (TA)$. The measure μ is quasi-invariant relative to shifts $x \in A' \subset (TA)$ and relative to left (or right) multiplication on $(1+y) \in A'$, where $y \in A'$, then this implies that 1 is the compact operator on A (see also § 3.24). This is possible only when A is finite dimensional over $\mathbf{K}$.

3.29. Theorem. *Let A be a Banach space over a locally compact infinite field $\mathbf{K}$ supplied with a non-trivial non-Archimedean multiplicative norm. If μ is a non-trivial real-valued measure on $Bf(A)$ quasi-invariant relative to shifts from a dense $\mathbf{K}$-linear subspace L' in A, then there exists a nontrivial topological group G of $\mathbf{K}$-linear automorphisms of A such that μ is also quasi-invariant relative to G.*

Proof. Take G consisting of $\mathbf{K}$-linear operators satisfying conditions of Theorem 3.24. They form a multiplicative group, since the conditions of Theorem 3.24 are satisfied for the product UV of two operators U and V satisfying these conditions, also $det(UV) = det(U)det(V)$ for each pair of operators in G and $\tilde{\rho}$ satisfies the co-cycle condition. In the topology of G inherited from the Banach space $L(A)$ of bounded $\mathbf{K}$-linear operators this group G is the topological group.

1.4. Pseudo-differentiable Measures

4.1. Definition and notes. A function $f : \mathbf{K} \to \mathbf{R}$ is called pseudo-differentiable of order b, if there exists the following integral:

$$PD(b, f(x)) := \int_{\mathbf{K}} [(f(x) - f(y)) \times g(x,y,b)]dv(y). \qquad (1)$$

We introduce the following notation $PD_c(b, f(x))$ for such integral by $B(\mathbf{K}, 0, 1)$ instead of the entire $\mathbf{K}$. Where $g(x,y,b) :=| x-y |^{-1-b}$ with the corresponding Haar measure v with values in $\mathbf{R}$, where $b \in \mathbf{C}$ and $|x|_{\mathbf{K}} = p^{-ord_p(x)}$.

Obviously, the definitions of differentiability of measures can not be transferred from [BS90, DF91] onto the case considered here. This is the reason why the notion of pseudo-differentiability is introduced here. A quasi-invariant measure μ on X is called pseudo-differentiable for $b \in \mathbf{C}$, if there exists $PD(b, g(x))$ for $g(x) := \mu(-xz + S)$ for each $S \in Af(X, \mu)$ with $|m|(S) < \infty$ and each $z \in J^b_\mu$, where J^b_μ is a $\mathbf{K}$-linear subspace dense in X. For a fixed $z \in X$ such measure is called pseudo-differentiable along z.

For a one-parameter subfamily of operators $B(\mathbf{K}, 0, 1) \ni t \to U_t : X \to X$ quasi-invariant measure μ is called pseudo-differentiable for $b \in \mathbf{C}$, if for each S the same as above there exists $PD_c(b, g(t))$ for a function $g(t) := \mu(U_t^{-1}(S)$, where X may be also a topological group G with a measure quasi-invariant relative to a dense subgroup G' (see [Lud99t, Lud98s, Lud00a]).

4.2. Let μ, X, and ρ be the same as in Theorem 3.24 and F be a non-Archimedean Fourier transform defined in [VVZ94, Roo78].

Theorems. (1) *$g(t) := \rho(z+tw, x)j(t) \in L^1(v, K \to \mathbf{C}) := V$, moreover, there exists $F(g) \in C^0(\mathbf{K}, \mathbf{C})$ and $\lim_{|t|\to\infty} F(g)(t) = 0$ for μ and v with values in $\mathbf{R}$, where z and $w \in J_\mu$, $t \in \mathbf{K}$, $j(t)$ is the characteristic function of a compact subset $W \subset \mathbf{K}$. In general, may be $k(t) := \rho(z+tw, x) \notin V$.*

(2) *Let $g(t) = \rho(z+tw, x)j(t)$ with clopen subsets W in $\mathbf{K}$. Then there are μ, for which there exists $PD(b, g(t))$ for each $b \in \mathbf{C}$. If $g(t) = \rho(z+tw, x)$, then there are probability measures μ, for which there exists $PD(b, g(t))$ for each b with $0 < Re(b)$ or $b = 0$.*

(3) *Let* $S \in Af(X,y)$, $|\mu|(S) < \infty$, *then for each* $b \in U := \{b' : Re\ b' > 0$ *or* $b' = 0\}$ *there is a pseudo-differentiable quasi-invariant measure* μ *.*

Proof. We consider the following additive compact subgroup $G_T := \{x \in X | \|x(j)\| \le p^{k(j)}$ for each $j \in \mathbf{N}\}$ in X, where $T = diag\{d(j) \in K : |d(j)| = p^{-k(j)}$ for each $j \in \mathbf{N}\}$ is a compact diagonal operator. Then μ from Theorem 3.20 is quasi-invariant relative to the following additive subgroup $S_T := G_T + H$, where $H := span_{\mathbf{K}}\{e(j) : j \in \mathbf{N}\}$. Moreover, for each z and $w \in G_T$ and $R > 0$ there is $c > 0$ such that $\rho(z + uw, x) = \rho(z + sw, x)$ for $|u - s| < c$ and $x \in B(X, 0, R)$, if all functions f_j in the proof of Theorem 3.20 are locally constant (that is, f are defined on $\mathbf{K}e_j \subset X$). In general, for each $b \in \mathbf{C}$ we can choose a sequence $h_j(x)$ with $\sum_{j=0}^{\infty} \sup_{x \in \mathbf{K}}(|h_j(x)/(h_j(x) + f_j(x))| r_j(x)) < c'$ for suitable fixed $c' > 0$ and functions $r_j : \mathbf{K} \to [0, \infty)$ with $\lim_{|x| \to 0} r_j(x) = \infty$. Carrying out calculations and using the fact that μ is the probability quasi-invariant measure we get the pseudo-differentiability of μ. Using the Riemann-Lebesgue theorem (see it in [VVZ94]) we get the statement of this theorem for $F(g)$.

4.2.1. Theorem. *Let* X *be a Banach space of separable type over a locally compact non-Archimedean field* $\mathbf{K}$ *and* J *be a dense proper* $\mathbf{K}$*-linear subspace in* X *such that the embedding operator* $T : J \hookrightarrow X$ *is compact and nondegenerate,* $ker(T) = \{0\}$, $b \in \mathbf{C}$. *Then a set* $\mathcal{P}_b(X,J)$ *of probability measures* μ *on* $Bf(X)$ *quasi-invariant and pseudo-differentiable of order* b *relative to* J *is of cardinality* 2^{c}. *If* J', $J' \subset J$, *is also a dense* $\mathbf{K}$*-linear subspace in* X, *then* $\mathcal{P}_b(X,J') \supset \mathcal{P}_b(X,J)$.

Proof. As in § 3.20.1 choose for T an orthonormal base in X in which T is diagonal and X is isomorphic with c_0 over $\mathbf{K}$ such that in its standard base $\{e_j : j \in \mathbf{N}\}$ the operator T has the form $Te_j = a_j e_j$, $0 \ne a_j \in \mathbf{K}$ for each $j \in \mathbf{N}$, $\lim_{j \to \infty} a_j = 0$ (see Appendix). Take g_n from § 3.20.1, where $g_n \in L^1(\mathbf{K}, Bf(\mathbf{K}), \nu(dx/a_n), \mathbf{R})$, satisfy conditions there and such that there exists $\lim_{m \to \infty} PD(b, \prod_{n=1}^{m} g_n(xz)) \in L^1(X, Bf(X), \nu, \mathbf{C})$ by the variable x for each $z \in J$, where $x \in \mathbf{K}$. Evidently, $\mathcal{P}(X,J) \subset \mathcal{M}(X,J)$. The family of such sequences of functions $\{g_n : n \in \mathbf{N}\}$ has the cardinality 2^{c}, since in $L^1(\nu)$ the subspace of step functions is dense and the condition of pseudo-differentiability is the integral convergence condition (see §§ 4.1 and 4.2). The latter statement of this theorem is evident, since the family of all $\{g_n : n\}$ satisfying conditions above for J also satisfies such conditions for J'.

4.3. Let X be a Banach space over $\mathbf{K}$, $b_0 \in \mathbf{R} \cup \{+\infty\}$ and suppose that the following conditions are satisfied:

(1) $T : X \to X$ is a compact operator with $ker(T) = \{0\}$;

(2) a mapping $\tilde{F}$ from $B(\mathbf{K}, 0, 1)$ to $C_T(X) := \{U : U \in C^1(X,X)$ and $(U'(x) - I)$ is a compact operator for each $x \in X$, there is U^{-1} satisfying the same conditions as $U\}$ is given;

(3) $\tilde{F}(t) = U_t(x)$ and $\bar{\Phi}^1 U_t(x + h, x)$ are continuous by t, that is, $\tilde{F} \in C^1(B(\mathbf{K}, 0, 1), C_T(X))$;

(4) there is $c > 0$ such that $\|U_t(x) - U_s(x)\| \le \|Tx\|$ for each $x \in X$ and $|t - s| < c$;

(5) for each $R > 0$ there is a finite-dimensional over $\mathbf{K}$ subspace $H \subset X$ and $c' > 0$ such that $\|U_t(x) - U_s(x)\| \le \|Tx\|/R$ for each $x \in X \ominus H$ and $|t - s| < c'$ with $(3 - 5)$ satisfying also for U_t^{-1}.

Theorem. *On* X *there are probability quasi-invariant measures* μ *which are pseudo-differentiable for each* $b \in \mathbf{C}$ *with* $\mathbf{R} \ni Re(b) \le b_0$ *relative to a family* U_t.

Proof. From Conditions (2,3) it follows that there is $c > 0$ such that $|det(U_t'(x))| = |det(U_s'(x))|$ for μ-a.e. $x \in X$ and all $|t - s| < c$, where quasi-invariant and pseudo-differentiable measures μ on X relative to S_T may be constructed as in the proof of Theorems 3.20 and 4.2. From $(1-5)$ it follows that conditions of Theorem 3.25 are satisfied for each U_t. From $(3,5)$ it follows that for each $R>0$ and $b>0$ there exists $c'>0$ such that $|\rho_\mu(x-U_t^{-1}(x),x)-\rho_\mu(x-U_t^{-1}(x),x)|\le b$ for each $|t-s|<c'$ and $x\in B(X,0,R)$. If during construction of μ to use only locally constant $f_j(x_j)$ with $h_j=0$, then we can take $\infty\ge b_0\ge 0$. Indeed, let $T=SCDE$ be a decomposition from the appendix and $\mu^A(\tilde{S})=\mu(E\tilde{S})$ with $A=E^{-1}$ (or C), $\tilde{S}\in Bf(X)$, then $\rho_{\mu^A}(x-U_t^{-1}(x),x)=\rho_\mu(Ex-U_t^{-1}(Ex),Ex)$ and $c(x):=\|U_t^{-1}(Ex)-U_s^{-1}(Ex)\|\le\|TEx\|$ for $x\in X$ and $c(x)\le\|TEx\|/R$ for $x\in X\ominus H$ (for each $R>0$ there is $H\subset X$ with $dim_{\mathbf{K}}H<\infty$), due to compactness of $E-I$ and $C-I$ we can choose H such that $A^{-1}H=H$, $AH=H$. We use also images of quasi-invariant measure constructed with the help of compact diagonal operator $(D'-I)$, where $|D_j-1|<|D'_j-1|p^{k(j)}$, $\lim_j k(j)=-\infty$. Then $\sup\{\|U_t(x)-U_t(0)\|:x\in B(X,0,R)\}\le\|\bar{\Phi}^1U_t\|_{B(X,0,R)}\times\|x\|$. From the fact that H is finite-dimensional and the existence of the orthogonal projector (that is, corresponding to the decomposition into the direct sum) $\pi_H:X\to H$ it follows that $\|U_t^a(e_j)-U_s^a(e_j)\|\le b_j$ for each $j\in\mathbf{N}$ and each $|t-s|<c'$, where $\lim_{j\to\infty}b_j=0$, $a=1$ or $a=-1$. This guarantees pseudo-differentiability of μ.

4.4. Let X be a Banach space of separable type over $\mathbf{K}$, μ be a probability quasi-invariant measure $\mu:Bf(X)\to\mathbf{R}$, that is pseudo-differentiable for a given r with $Re(r)>0$, $C_b(X)$ be a space of continuous bounded functions $f:X\to\mathbf{R}$ with $\|f\|:=\sup_{x\in X}|f(x)|$.

Theorem. *For each $a\in J_\mu$ and $f\in C_b(X)$ is defined the following integral:*

$$(i)\ l(f)=\int_{\mathbf{K}}\left[\int_X f(x)[\mu(-\lambda a+dx)-\mu(dx)]\right]g(\lambda,0,r)v(d\lambda)$$

and there exists a measure $\nu:Bf(X)\to\mathbf{C}$ with a bounded variation. For $0<r\in\mathbf{R}$ this ν is a mapping from $Bf(X)$ into $\mathbf{R}$ such that

$$(ii)\ l(f)=\int_X f(x)\nu(dx),$$

where v is the Haar measure on $\mathbf{K}$ with values in $[0,\infty)$, moreover, ν is independent from f and may be dependent on $a\in J_\mu$. We denote $\nu=:\tilde{D}_a^r\mu$.

Proof. From Definition 4.1 and the Fubini and Lebesgue theorems it follows that there exists

$$\lim_{j\to\infty}\int_{\mathbf{K}\setminus B(\mathbf{K},0,p^{-j})}\left[\int_X(f(x+\lambda a)-f(x))g(\lambda,0,r)\mu(dx)\right]v(d\lambda)=l(f),$$

that is (i) exists. Let

$$(iii)\quad l_j(V,f):=\int_{\mathbf{K}\setminus B(\mathbf{K},0,p^{-j})}\left[\int_V f(x)(\mu(-\lambda a+dx)-\mu(dx))g(\lambda,0,r)\right]v(d\lambda),$$

where $V\in Bf(X)$.

Consider the measure $\nu_{\lambda,a}:=(\mu_{\lambda a}-\mu)g(\lambda,0,r)$ for $a\in J_\mu$, $\lambda\in\mathbf{K}$.

Let B be some subset in X and B^h be an h-enlargement of B so that $B^h:=\{x\in X:\inf_{y\in B}\|x-y\|\le h\}$, where $h>0$. We show that for each $b>0$ there exists a compact C such that $|\nu_{\lambda,a}|(X\setminus C)<b$ for each $\lambda\ne 0$, where $|\nu|$ denotes the variation of a real-valued

measure ν. For demonstrating this it is sufficient to prove that for each $b>0$ and $h>0$ there exists a compact C_1 so that $|\nu_{\lambda,a}|(X\setminus C_1^h)<b$ for each $\lambda\neq 0$. Indeed, if the latter is satisfied, then choose $h_n>0$ monotonously decreasing and converging to zero with n tending to the infinity and take a compact C_n so that $|\nu_{\lambda,a}|(X\setminus C_n^{h_n})<b/2^n$ for each $\lambda\neq 0$. Then $C=\bigcap_{n=1}^{\infty}C_n^{h_n}$ is compact, since it is closed and C has a finite h_n-net for each n. Moreover, $|\nu_{\lambda,a}|(X\setminus C)\le\sum_{n=1}^{\infty}|\nu_{\lambda,a}|(X\setminus C_n^{h_n})\le\sum_{n=1}^{\infty}b/2^n=b$ for each $\lambda\in\mathbf{K}\setminus\{0\}$.

Suppose now the contrary that for some $b>0$ and $h>0$ there does not exist any compact C so that $|\nu_{\lambda,a}|(X\setminus C^h)<b$ for all $\lambda\in\mathbf{K}\setminus\{0\}$. The field $\mathbf{K}$ is locally compact and separable, the Banach space X is of separable type over $\mathbf{K}$, hence X is the Radon space, since X is the separable complete metric space. Therefore, each measure $|\nu_{\lambda,a}|$ is Radon for $\lambda\neq 0$.

Choose a sequence of compacts C_n and $\lambda_n\neq 0$ with $|\nu_{\lambda_n,a}|(X\setminus C_n)>b/2$, $|\nu_{\lambda_n,a}|(X\setminus\bigcup_{j=1}^{n}C_j)\le b/8$ with $C_n\subset X\setminus\bigcup_{j=1}^{n-1}C_j^h$ for each $n\in\mathbf{N}$. The norm in X is non-Archimedean, hence the sets C_n^h are pairwise disjoint for different n, since $C_n\times C_j$ is compact and a continuous real-valued function on a compact achieves at some point in it its infimum, so that $\inf_{y\in C_n,x\in C_j}\|x-y\|>h$ for each $1\le j<n$.

For each n there exists a continuous function $f_n(x)$ equal to zero on $X\setminus C_n^h$ and such that $\int_X f_n(x)\nu_{\lambda_n,a}(dx)\ge b/2$ and $\sup_{x\in X}|f_n(x)|\le 1$.

For each subsequence $\{n_k:k\}$ the series $\sum_k f_{n_k}(x)$ converges and is the continuous function with the norm $\|f\|_{C_b^0}:=\sup_{x\in X}|f(x)|\le 1$.

For each prime number $s>1$ we construct a function $g_s(x):=\sum_j f_{s^{n_j}}(x)$, where a finite or infinite set $\{n_j:j\}$ is chosen so that for each prime number $s>1$ the inequality

$$\left|\int_X\sum_j f_{s^{n_j}}(x)\nu_{\lambda_{s^m},a}(dx)\right|>b/8 \qquad (1)$$

for some $m>n_k$ is satisfied. We describe the inductive procedure of choice to achieve this.

Take $n_1=1$ and if $n_1<n_2<\dots<n_k$ are chosen, then put

$$n_{k+1}:=\inf\left\{m:m>n_k,\ \text{and}\left|\int_X\left(\sum_{j=1}^{k}f_{s^{n_j}}(x)+f_{p^m}(x)\right)\nu_{\lambda_{s^m},a}(dx)\right|\ge b/4\right\}. \qquad (2)$$

If the set in the curled brackets is is void, then $\{n_1,\dots,n_k\}$ is the desired set of indices. In accordance with the definition of n_{k+1} for $n_k<m<n_{k+1}$ the inequality

$$\left|\int_X\left(\sum_{j=1}^{k}f_{s^{n_j}}(x)+f_{p^m}(x)\right)\nu_{\lambda_{s^m},a}(dx)\right|<b/4$$

is fulfilled. Therefore,

$$\left|\int_X\sum_{j=1}^{k}f_{s^{n_j}}(x)\nu_{\lambda_{s^m},a}(dx)\right|\ge b/4 \qquad (3)$$

for each m such that either $n_k<m\le n_{k+1}$ when n_{k+1} exists or $n_k<m$ if n_{k+1} does not exist. Thus

$$\left|\int_X\sum_j f_{s^{n_j}}(x)\nu_{\lambda_{s^m},a}(dx)\right|$$

$$\geq \left| \int_X \sum_{n_j \leq m} f_{s^{n_j}}(x) \nu_{\lambda_{s^m},a}(dx) \right| - \left| \int_X \sum_{n_j > m} f_{s^{n_j}}(x) \nu_{\lambda_{s^m},a}(dx) \right|$$

$$\geq b/4 - |\nu_{\lambda_{s^m},a}| \Big(X \setminus \bigcup_{l=1}^{s^m} C_l \Big) \geq b/8$$

and the desired set $\{n_k : k\}$ is constructed together with the function $g_s(x)$.

We have that $\lim_{|\lambda| \to \infty} |\nu_{\lambda,a}|(X) = 0$, since $r > 0$, $|\mu|(X) < \infty$, $|\mu_{\lambda a}|(X) = |\mu|(X)$, while $\lim |\lambda| \to \infty g(\lambda, 0, r) = 0$ for $r > 0$. Therefore, due to $|\nu_{\lambda_n,a}|(X) \geq b$, where $b > 0$, we can without loss of generality take a bounded sequence of λ_n. Then due to local compactness of the field **K** this sequence has an accumulation point. So we can consider the case when the sequence λ_n converges with n tending to the infinity. Since

$$\lim_{n\to\infty} \int_X g_s(x) \nu_{\lambda_n,a}(dx) = \lim_{n\to\infty} \int_X g_s(x) \nu_{\lambda_{s^m},a}(dx),$$

then

$$\left| \lim_{n\to\infty} \int_X g_s(x) \nu_{\lambda_n,a}(dx) \right| \geq b/8.$$

Choose $0 < b_s \leq 1$ such that

$$\lim_{n\to\infty} \int_X b_s g_s(x) \nu_{\lambda_n,a}(dx) \geq b/8,$$

then

$$\lim_{n\to\infty} \int_X \sum_{s=2}^N b_s g_s(x) \nu_{\lambda_n,a}(dx) \geq k(N) b/8,$$

where $k(N)$ denotes the number of prime numbers in $[2,N]$. But $\| \sum_{s=2}^N b_s g_s(x) \|_{C^0_b} \leq 1$ for each N, since $\|g_s\|_{C^0_b} \leq 1$ for each s, while for different s the functions g_s can not be sumultaneously non-zero.

The functional $l(f)$ is the limit of the convergent sequence of **C**-linear and continuous functionals on the space of continuous bounded complex-valued functions $f : X \to \mathbf{C}$, hence $l(f)$ is also **C**-linear and continuous. This means that there exists a constant $y > 0$ with $|l(f)| \leq y \|f\|_{C^0_b}$ for each $f \in C^0_b(X, \mathbf{C})$. This implies that

$$k(N) b/8 \leq \lim_{n\to\infty} \int_X \sum_{s=2}^N b_s g_s(x) \nu_{\lambda_n,a}(dx) \leq y, \tag{4}$$

but this is impossible for $k(N) > 8y/b$, since $\lim_{N\to\infty} k(N) = \infty$. This contradiction demonstrates that the claimed above compact set exists.

Hence for each $c > 0$ there exists a compact $V_c \subset X$ with $|\nu_\lambda|(X \setminus V_c) < c$ for each $|\lambda| > 0$, where $\nu_\lambda(A) := \int_{\mathbf{K} \setminus B(\mathbf{K},0,|\lambda|)} [\mu(-\lambda' a + A) - \mu(A)] g(\lambda', 0, b) v(d\lambda')$ for $A \in Bf(X)$. Also there are $\delta > 0$ and V_c such that $-\lambda' a + V_c \subset V_c$ and $\| [X \setminus V_c) \triangle (-\lambda' a + (X \setminus V_c))] \|_\mu = 0$ for each $|\lambda'| < \delta$ (see also Theorem 7.22 [Roo78]), where $A \triangle B := (A \setminus B) \cup (B \setminus A)$.

Each continuous linear functional on $C_b(V)$ for compact V has the form (ii) (see [Bou63-69], A.5 [Sch84]), that is, for $l_j(V, f)$ there exists a measure $\nu_\lambda(c, *)$. In view of the Alaoglu-Bourbaki theorem each bounded subset W in $[C_b(V_c)]'$ is precompact (i.e., $cl(W)$ is compact) in the weak topology, where $C_b(V_c)$ is separable. For each $c>0$ we have

$$l(f)=\int_{V_c} f(x)\nu(c,dx)+O(c\times\sup_{x\notin V_c}|f(x)|)$$

such that $\nu(c,*)$ are measures on $Bf(V_c)$. Choosing $V_c\subset V_{c'}$ for $c>c'$ and defining $\nu(A):=\lim_{c\to+0}\nu(c,A\cap V_c)$ for each $A\in Bf(X)$ in view of the Radon-Nikodym theorem about convergence of measures (see [Con84]) we get (i) with the help of the theorem about extension of measures, since the minimal σ-field $\sigma Bco(X)$ generated by $Bco(X)$ coincides with $Bf(X)$.

4.5. Theorem. *If μ is probability real-valued quasi-invariant measure on a Banach space X over $\mathbf{K}=\mathbf{Q_p}$, $a\in J_\mu$ and μ is pseudo-differentiable of order* 1, *then for each $f\in L^\infty(X,\mu,\mathbf{R})$ with $supp(f)\subset B(X,0,R)$, $\infty>R=R(f)>0$ the following equality is accomplished:*

$$(i)\ \int_X[D_\lambda^{-1}f(x+\lambda a)]|_{\lambda=0}^{\lambda=\beta}(\tilde{D}_a^1\mu)(dx)\ =\int_X[f(x+\beta a)-f(x)]\mu(dx),$$

moreover, pseudo-differential $\tilde{D}_a^1\mu:=\nu$ in the direction a from Theorem 4.4 *given by formula* (ii) *characterizes μ uniquely.*

Proof. From the restrictions on f it follows that $f(x+\lambda a)=:\psi(\lambda)\in\mathsf{E}'$, where E' is the topologically dual space to the space E of basic locally constant functions [VVZ94]. By formula (1.5) § 9[VVZ94] for them there is the following equality $D^aD^b\psi=D^bD^a\psi=D^{a+b}\psi$ for $a+b\neq-1$, $a,b\in\mathbf{R}$ (D_λ^a denotes a pseudo-differentiation of functions by λ). In view of Theorem in § 6.3[VVZ94] $\mathsf{E}'=[\phi\in\mathsf{D}':supp(\phi)$ is compact], where $\mathsf{D}=[\phi\in\mathsf{E}:supp(\phi)$ is compact]. Since $[\xi]|_0^\beta:=\xi(\beta)-\xi(0)$ for $\xi(\lambda)=D_\lambda^{-1}\psi(\lambda)$ and $(D^{-1}D\psi)(\lambda)=\psi(\lambda)$, hence due to the definition of D_λ^{-1} we have:

$$p^2(p+1)^{-1}\int_X[D_\lambda^{-1}f(x+\lambda a)]|_0^\beta(\tilde{D}_a^1\mu)(dx)=\lim_{\alpha\to-1}\int_{\mathbf{Q_p}}\int_{\mathbf{Q_p}}(1-p^\alpha)$$

$$(1-p^{-1-\alpha})^{-1}p^2(1+p)^{-1}\left[\int_X(\mu(-a(\lambda'+\lambda")+dx)-\mu(-a\lambda'+dx))\right.$$

$$\left.|\lambda"|_p^{-2}|\lambda-\lambda'|_p^{-1-\alpha}\mu(dx)\right]\nu(d\lambda')\nu(d\lambda").$$

In view of the Fubini and Lebesgue theorems we establish Formula (i). Taking all different $f\in L^\infty(X,\mu,\mathbf{R})$ such that $\psi(\lambda)=f(x+a\lambda)$ are locally polynomial by λ and using the non-Archimedean variant of the Stone-Weiestrass theorem by Kaplanski [Sch84] we get that μ is characterized by the left side of (i) uniquely.

4.6. (Properties of $\tilde{D}_a^1\mu$ for real-valued measures μ (see also §§ 4.4. and 4.5), where $\mathbf{K}=\mathbf{Q_p}$.)

Proposition.1. *If μ is pseudo-differentiable in a direction $a\in J_\mu^1$, $a\neq0$, then for each $d\in J_\mu^1$ a measure $\mu^d(A):=\mu(A-d)$, $A\in Bf(X)$, is also pseudo-differentiable in a direction a, moreover, $\tilde{D}_a^1(\mu^d)=(\tilde{D}_a^1\mu)_d$. If $a\in J_\mu^b$, then $\tilde{D}_{\lambda a}^b\mu$ exists for each $\lambda\in\mathbf{K}$.*

Proof. From §§ 4.4 and 4.5 it follows that

$$\int_X f(x)(\tilde{D}_a^{-1}\mu)^d(dx) = \int_{\mathbf{K}}\int_X [f(x+\lambda a)-f(x)]|\lambda|_{\mathbf{K}}^{-2}\mu^d(dx)\nu(d\lambda)$$

$$= \int_X f(x)(\tilde{D}_a^1(\mu^d))(dx).$$

The last statement follows from Definition 4.1 with the help of § 3.21.

Proposition.2. *If* $a\in J_\mu^1$, $\nu=\tilde{D}_a^1\mu$ *and* $D_\lambda^{-1}\rho_\nu(\lambda a,x) =: \phi(x)\in L^1(X,\nu,\mathbf{R})$ *by x, then* $\mu^{\lambda a}\ll\nu$.

Proof. In view of Formula 4.5.(i) the following equality is accomplished:

$$\int_X f(x)\mu^{\lambda a}(dx) = \int_X f(x)\mu(dx) + \int_X f(x)[D_\xi^{-1}\rho_\nu(\lambda a,x)]|_{\xi=0}^{\xi=\lambda}\nu(dx).$$

Taking $f(x)=\chi_A(x)$, where χ_A is a characteristic function of $A\in Bf(X)$ we get $\mu^{\lambda a}(A)=\mu(A)$ for every $|\lambda|>0$. Therefore, considering $|\lambda|\to\infty$ we get $\mu(A)=0$, that is, $\mu\ll\nu$ and $\mu^{\lambda a}\ll\nu$.

3. If a non-negative measure μ is pseudo-differentiable in a direction $a\in J_\mu^r$ and $(\tilde{D}_a^r\mu)\ll\mu$, then $l_\mu^r(a,x) := (\tilde{D}_a^r\mu)(dx)/\mu(dx)$ is called the logarithmic pseudo-derivative of quasi-invariant measure μ in a direction a of order r, where $r>0$. Define the sequence of logarithmic pseudo-derivatives l_n^r of μ_n, where μ_n denotes the projection of the measure μ on finite dimensional over $\mathbf{K}$ subspaces $H(n)$, $P_n : X\to H(n)$ is the projection, $H(n)\subset H(n+1)$ for each $n\in\mathbf{N}$, $\bigcup_n H(n)$ is everywhere dense in X.

Theorem. *The sequence of functions* $l_n^r(P_na,P_nx)$ *is the martingale. The logarithmic pseudo-derivative* $l_\mu^r(a,x)$ *in a direction* $a\in X$ *of order* $r>0$ *exists if and only if the sequence* $l_n^r(P_na,P_nx)$ *is uniformly integrable. Moreover,*

$$l_\mu^r(a,x) = \lim_{n\to\infty} l_n^r(P_na,P_nx) \quad (mod\ \mu) \text{ and for } r=1 \text{ there is the equality:} \tag{1}$$

$$\int_X [f(x+\lambda a)-f(x)]\mu(dx) = \int_X \left[\tilde{D}_\xi^{-1}f(x+\xi a)\right]|_{\xi=0}^{\xi=\lambda} l_\mu(a,x)\mu(dx). \tag{2}$$

Proof. Let μ be pseudo-differentiable along a_1 and a_2. Evidently, $g(x,y,r)=g(y,x,r)=g(x-y,0,r)$ for all $x\ne y$ (see § 4.1). In view of 4.4(i,ii) we have

$$\int_X f(x)\tilde{D}_{a_1+a_2}^r\mu(dx)$$

$$= \int_{\mathbf{K}}\left[\int_X f(x)[\mu(-\lambda(a_1+a_2)+dx)-\mu(dx)]\right]g(\lambda,0,r)\nu(d\lambda)$$

$$= \int_{\mathbf{K}}\left[\int_X [f(x+\lambda(a_1+a_2))-f(x)]\mu(dx)\right]g(\lambda,0,r)\nu(d\lambda)$$

$$= \int_{\mathbf{K}}\left[\int_X [f(x+\lambda(a_1+a_2))-f(x+\lambda a_1)]\mu(dx)\right]g(\lambda,0,r)\nu(d\lambda)$$

$$+ \int_{\mathbf{K}}\left[\int_X [f(x+\lambda a_1)-f(x)]\mu(dx)\right]g(\lambda,0,r)\nu(d\lambda). \tag{3}$$

The Banach space X is totally disconnected. The integrals on the right side exist for each continuous simple function f, since $\chi_B(x+y)=\chi_B(x)$ for each clopen ball $B(X,x_0,R):=\{x\in X:|x-x_0|\le R\}$ of radius $R>0$ with $|y|<R$, where χ_A denotes the characteristic function of a set A, $\chi_A(x)=1$ for each $x\in A$, $\chi_A(x)=0$ for each $x\notin A$. The space of continuous simple functions is dense in $C^0_b(X,\mathbf{C})\cap L^1(\mu)$, consequently, (3) has the continuous extension to the $\mathbf{C}$-linear functional on $C^0_b(X,\mathbf{C})$. Thus $\tilde{D}^r_{a_1+a_2}\mu$ exists.

Suppose that $\tilde{D}^r_a\mu$ exists and $\xi\in\mathbf{K}$, $\xi\ne 0$, then

$$\int_X f(x)\tilde{D}^r_{\xi a}\mu(dx)=\int_{\mathbf{K}}\left[\int_X f(x)\left[\mu(-\lambda\xi a+dx)-\mu(dx)\right]\right]g(\lambda,0,r)v(d\lambda) \tag{4}$$

$$=\int_{\mathbf{K}}\left[\int_X[f(x+\lambda\xi a)-f(x)]\mu(dx)\right]g(\lambda,0,r)v(d\lambda)$$

$$=[1/mod_{\mathbf{K}}(\xi)]\int_{\mathbf{K}}\left[\int_X[f(x+za)-f(x)]\mu(dx)\right]g(\lambda,0,r)v(dz)$$

$$=[1/mod_{\mathbf{K}}(\xi)]\int_X f(x)\tilde{D}^r_a\mu(dx),$$

hence $mod_{\mathbf{K}}(\xi)\tilde{D}^r_{\xi a}\mu=\tilde{D}^r_a\mu$ exists for each $0\ne\xi\in\mathbf{K}$, where $\tilde{D}^r_0\mu=0$.

Thus we have demonstrated, that the set X^r_μ of all vectors a in X for which $\tilde{D}^r_a\mu$ exists is the $\mathbf{K}$-linear subspace in X. If $l^r_\mu(a,x)$ exists, then

$$\int_X f(x)\tilde{D}^r_a\mu(dx)=\int_{\mathbf{K}}\left[\int_X[f(x+\lambda a)-f(x)]\mu(dx)\right]g(\lambda,0,r)v(d\lambda)$$

$$=\int_X f(x)l^r_\mu(a,x)\mu(dx). \tag{5}$$

This implies that the set L^r_μ of all those $a\in X$ for which the logarithmic pseudo-derivatives of order r exists is the $\mathbf{K}$-linear subspace in X. Particularly, if $f(x)=\phi(P_Lx)$ is a cylindrical function, where $P_L:X\to L$ is the projection operator, for example, for $L=H(n)$, $a\in L^r_\mu$ and $z=P_La$, then

$$\int_{\mathbf{K}}\left[\int_L[\phi(x+zu)-\phi(x)]\mu_L(dx)\right]g(z,0,r)v(dz)$$

$$=\int_{\mathbf{K}}\left[\int_X[f(x+zu)-f(x)]\mu(dx)\right]g(z,0,r)v(dz)$$

$$=\int_X f(x)\tilde{D}^r_a\mu(dx)=\int_L\phi(x)\tilde{D}^r_z\mu_L(dx), \tag{6}$$

hence $\tilde{D}^r_z\mu_L(dx)\ll\mu_L$ and inevitably μ_L has the logarithmic pseudo-derivative along z of order $r>0$.

In view of Theorem 3.2.4 we get:

$$d(\tilde{D}^r_z\mu_L(x))/d\mu_L(x)=\int_X l^r_\mu(a,z)\mu(dz,\mathcal{B}^L|P_L^{-1}x), \tag{7}$$

where $\mu(dz, \mathcal{B}^L|P_L^{-1}x)$ is the conditional measure corresponding to μ relative to the σ-algebra $\mathcal{B}^L$ of all sets $P_L^{-1}A$, where $A \in Bf(L)$ is a Borel set in L.

Consider the sequence of logarithmic pseudo-derivatives l_n^r of μ_n, where μ_n denotes the projection of the measure μ on $H(n)$, $n \in \mathbf{N}$. In view of Formula (7)

$$d(\tilde{D}^r_{a(n)}\mu_{H(n)}(P_n x))/d\mu_{H(n)}(P_n x) = \int_{H(m)} l^r_\mu(a(m),z)\mu_{H(m)}(dz, \mathcal{B}^{H(n)}_{H(m)}|x) \tag{8}$$

for each $n < m$, where $\mu_{H(m)}(dz, \mathcal{B}^{H(n)}_{H(m)}|x)$ is the conditional measure corresponding to $\mu_{H(m)}$ relative to the σ-algebra $\mathcal{B}^{H(n)}_{H(m)}$ of all sets $H(m) \cap P_n^{-1}A$, where $A \in Bf(H(m))$ is a Borel set in $H(m)$, $P_n : X \to H(n)$ is the projection operator, $a(n) = P_n a$. Moreover, $\mu_{H(m)}(A, \mathcal{B}^{H(n)}_{H(m)}|x) = \mu(P_m^{-1}A, \mathcal{B}^{H(m)}|P_m^{-1}x)$. Therefore, (8) takes the form

$$l_n^r(a(n),x) = \int_{H(m)} l_m^r(a(m),z)\mu(dz, \mathcal{B}^{H(n)}|x). \tag{9}$$

The function l_n^r is $\mathcal{B}^{H(n)}$-measurable, hence due to §§ 2.37 and 2.39

$$\int_X l_n^r(a(n),P_n x)\psi(x)\mu(dx) = \int_X \int_{H(m)} l_m^r(a(m),z)\mu_{H(m)}(dz, \mathcal{B}^{H(n)}_{H(m)}|x)\psi(x)\mu(dx)$$

$$= \int_{H(m)} l_m^r(a(m),P_m x))\psi(x)\mu(dx)$$

for each $\mathcal{B}^{H(n)}$-measurable function $\psi(x)$, where $x(m) = P_m x$. This means that l_n^r is the martingale. In accordance with Formula (7) we have

$$l_n^r(a(n),P_n x) = \int_X l_\mu^r(a,z)\mu(dz, \mathcal{B}^{H(n)}|x). \tag{10}$$

Consider the function $g_N(t)$ so that $g_N(t) = 0$ for $t < N$, $g_N(t) = t - N$ for reach $t \geq N$. To prove the uniform integrability it is sufficient to show that

$$\lim_{N\to\infty} \sup_n \int_X g_N(|l_n(a(n),P_n x)|)\mu(dx) = 0. \tag{11}$$

The function $g_N(t)$ is downward convex so by the Jensen inequality

$$g_N(|l_n^r(a(n),P_n x)|) \leq \int_X g_N(|l_\mu^r(a,z)|)\mu(dz, \mathcal{B}^{H(n)}|x), \text{ hence}$$

$$\int_X g_N(|l_n^r(a(n),P_n x)|)\mu(dx) \leq \int_X \int_X g_N(|l_\mu^r(a,z)|)\mu(dz, \mathcal{B}^{H(n)}|x)\mu(dx)$$

$$= \int_X g_N(|l_\mu^r(a,x)|)\mu(dx). \tag{12}$$

Since $g_N(t) \leq |t|$ for each $t \in \mathbf{R}$, $\int_X |l_\mu(a,x)|\mu(dx) < \infty$ and $\lim_{N\to\infty} g_N(|l_\mu^r(a,x)|) = 0$ for μ-almost all x, then $\lim_{N\to\infty} \int_X g_N(|l_\mu^r(a,x)|)\mu(dx) = 0$. Combining this with (12) we get (11). In view of Theorem 2.42 the limit $\lim_{n\to\infty} l_n^r(P_n a, P_n x) = l_\mu^r(a,x)$ exists for μ-almost all x.

Take an arbitrary continuous on $H(n)$ function ϕ and put $f(x)=\phi(P_n x)$, then due to $(5-7)$ we infer:

$$\int_X [f(x+a)-f(x)]\mu(dx)=\int_X\int_{\mathbf{K}} f(x+za)l_n^r(P_n a,P_n x)\nu(dz)\mu(dx). \qquad (13)$$

The uniform integrability of l_n permits to take the limit by n tending to the infinity, that gives the equality:

$$\int_X [f(x+a)-f(x)]\mu(dx)=\int_X\int_{\mathbf{K}} f(x+za)l_\mu^r(a,x)\nu(dz)\mu(dx). \qquad (14)$$

This functional is $\mathbf{C}$-linear and continuous by $f\in C_b^0(X,\mathbf{C})$, so (14) is satisfied for each $f\in C_b^0(X,\mathbf{C})$ and inevitably $l_\mu^r(a,x)=d((\tilde{D}_a^r\mu)/d\mu)(x)$. Applying §§ 4.2-4.4 we get Formula (2).

4.7. Theorem. *Let X be a Banach space over $\mathbf{K}$, $|*|=mod_{\mathbf{K}}(*)$ with a probability quasi-invariant measure $\mu: Bf(X)\to\mathbf{R}$ and it is satisfied Condition 3.21(i), suppose μ is pseudo-differentiable and*

$(viii)$ $J_b\mu\subset T"J_\mu$, $(U_t: t\in B(\mathbf{K},0,1))$ is a one-parameter family of operators such that Conditions 3.25$(i-vii)$ are satisfied with the substitution of J_μ onto J_μ^b uniformly by $t\in B(\mathbf{K},0,1)$, $J_\mu\supset T'X$, where $T',T":X\to X$ are compact operators, $ker(T')=ker(T")=0$. Moreover, suppose that there are sequences

(ix) $[k(i,j)]$ and $[k'(i,j)]$ with $i,j\in\mathbf{N}$, $\lim_{i+j\to\infty}k(i,j)=\lim_{i+j\to\infty}k'(i,j)=-\infty$ and $n\in\mathbf{N}$ such that $|T"_{i,j}-\delta_{i,j}|<|T'_{i,j}-\delta_{i,j}|p^{k(i,j)}$, $|U_{i,j}-\delta_{i,j}|<|T"_{i,j}-\delta_{i,j}|p^{k'(i,j)}$ and $|(U^{-1})_{i,j}-\delta_{i,j}|<|T"_{i,j}-\delta_{i,j}|p^{k'(i,j)}$ for each $i+j>n$, where $U_{i,j}=\tilde{e}_iU(e_j)$, $(e_j: j)$ is orthonormal basis in X. Then for each $f\in C_b(X)$ there is defined

$$(i)\ l(f)=\int_{B(K,0,1)}\left[\int_X f(x)[\mu(U_t^{-1}(dx))-\mu(dx)]\right]g(t,0,b)\nu(dt)$$

and there exists a measure $\nu: Bf(X)\to\mathbf{C}$ with a bounded total variation [particularly, for $b\in\mathbf{R}$ it is such that $\nu: Bf(X)\to\mathbf{R}$] and

$$(ii)\ l(f)=\int_X f(x)\nu(dx),$$

where ν is independent from f and may be dependent on $(U_t:t)$, $\nu=:\tilde{D}_{U_*}^b\mu$.

Proof. From the proof of Theorem 3.25 it follows that there exists a sequence $U_t^{(q)}$ of polygonal operators converging uniformly by $t\in B(\mathbf{K},0,1)$ to U_t and equicontinuously by indices of matrix elements in L^1. Then there exists $\lim_{q\to\infty}\lim_{j\to\infty}\int_{B(\mathbf{K},0,1)\setminus B(\mathbf{K},0,p^{-j})}[\int_X f(U_t^{-1}(x))-f(x)]g(t,0,b)\mu(dx)]\nu(dt)$ for each $f\in C_b(X)$. From conditions $(viii,ix)$, the Fubini and Lebesgue theorems it follows that for $\nu_\lambda:=\int_{B(\mathbf{K},0,1)\setminus B(\mathbf{K},0,|\lambda|)}[\mu(U_t^{-1}(A))-\mu(A)]g(t,0,b)\nu(dt)$ for $A\in Bf(X)$ for each $c>0$ there exists a compact $V_c\subset X$ and $\delta>0$ such that $|\nu_\lambda|(X\setminus V_c)<c$. Indeed, V_c and $\delta>0$ may be chosen due to pseudo-differentiability of μ, §§ 2.35, 3.9, 3.23, Formula (i), 3.21(i) and due to continuity and boundednessy (on $B(\mathbf{K},0,1)\ni t$) of $|det\, U_t'(U_t^{-1})(x))|_{\mathbf{K}}$ satisfying the following conditions $U_t^{-1}(V_c)\subset V_c$ and $\|(X\setminus V_c)\triangle(U_t^{-1}(X\setminus V_c))\|_\mu=0$ for each $|t|<\delta$, since $V_c=Y(j)\cap V_c$ are compact for every j. At the same time closed subsets $Y(j)\subset X$ for

$U_t^{(q)}$ may be chosen independent from t and in § 2.35 operator S_c may be chosen symmetric, $TX \supset J_\mu \supset T'X$, where T is a compact operator. Evidently, conditions of type (ix) are carried out for $V(j,x)$ and $U(j,x)$ uniformly by j. Repeating proofs 3.25 and 4.4 with the use of Lemma 2.5 for the family $(U_t : t)$ we get formulas (i,ii).

1.5. Convergence of Measures

Different types of convergence of measures were considered in [BS90, Con84, SF76, Top74, Top76]. The definitions and theorems given below are taking into account the properties of quasi-invariance and pseudo-differentiability of measures.

5.1. Definitions, notes and notations. Let S be a normal topological group with the small inductive dimension $ind(S) = 0$, S' be a dense subgroup, suppose their topologies are τ and τ' correspondingly, $\tau' \supset \tau|_{S'}$. Let G be an additive Hausdorff left-R-module, where R is a topological ring, $\mathsf{R} \supset Bf(S)$ be a σ-ring for real-valued measures, $\mathsf{M}(\mathsf{R},G)$ be a family of measures with values in G, $\mathsf{L}(\mathsf{R},G,R)$ be a family of quasi-invariant measure $\mu : \mathsf{R} \to G$ with $\rho_\mu(g,x) \times \mu(dx) := \mu^{g^{-1}}(dx) =: \mu(gdx)$, $R \times G \to G$ be a continuous left action of R on G such that $\rho_\mu(gh,x) = \rho_\mu(g,hx)\rho_\mu(h,x)$ for each $g,h \in S'$ and $x \in S$. Particularly, $1 = \rho_\mu(g,g^{-1}x)\rho_\mu(g^{-1},x)$, that is, $\rho_\mu(g,x) \in R_o$, where R_o is a multiplicative subgroup of R. Moreover, $zy \in \mathsf{L}$ for $z \in R_0$ with $\rho_{z\mu}(g,x) = z\rho_\mu(g,x)z^{-1}$ and $z \neq 0$. We suppose that topological characters and weights S and S' are countable and each open W in S' is precompact in S. Let $\mathbf{P}$" be a family of pseudo-metrics in G generating the initial uniformity such that for each $c > 0$ and $d \in \mathbf{P}$" and $\{U_n \in \mathsf{R} : n \in \mathbf{N}\}$ with $\cap\{U_n : n \in \mathbf{N}\} = \{x\}$ there is $m \in \mathbf{N}$ such that $d(\mu^g(U_n), \rho_\mu(g,x)\mu(U_n)) < cd(\mu(U_n),0)$ for each $n > m$, in addition, a limit ρ is independent μ-a.e. on the choice of $\{U_n : n\}$ for each $x \in S$ and $g \in S'$. Consider a subring $R' \subset \mathsf{R}$, $R' \supset Bf(S)$ such that $\cup\{A_n : n = 1,\dots,N\} \in R'$ for $A_n \in R'$ with $N \in \mathbf{N}$ and $S'R' = R'$. Then $\mathsf{L}(\mathsf{R},G,R;R') := \{(\mu,\rho_\mu(*,*)) \in \mathsf{L}(\mathsf{R},G,R) : \mu-\ R'-$ is regular and for each $s \in S$ there are $A_n \in R', n \in \mathbf{N}$ with $s = \cap(A_n : n), \{s\} \in R'\}$.

For pseudo-differentiable measures μ let $S" \subset S'$, $S"$ be a dense subgroup in S, $\tau'|S"$ is not stronger than $\tau"$ on $S"$ and there exists a neighborhood $\tau" \ni W" \ni e$ in which are dense elements lying on one-parameter subgroups $(U_t : t \in B(\mathbf{K},0,1))$. We suppose that μ is induced from the Banach space X over $\mathbf{K}$ due to a local homeomorphism of neighborhoods of e in S and 0 in X as for the case of groups of diffeomorphisms [Lud96] such that is accomplished Theorem 4.7 for each $U_* \subset S"$ inducing the corresponding transformations on X. In the following case $S = X$ we consider $S' = J_\mu$ and $S" = J_\mu^b$ with $Re(b) > 0$ such that $M_\mu \supset J_\mu \subset (T_\mu X)^\sim$, $J_\mu^b \subset (T_\mu^{(b)}X)^\sim$ with compact operators T_μ and $T_\mu^{(b)}$, $ker(T_\mu) = ker(T_\mu^{(b)}) = 0$ and norms induced by the Minkowski functional P_E for $E = T_\mu B(X,0,1)$ and $E = T_\mu^{(b)}B(X,0,1)$ respectively. We suppose further that for pseudo-differentiable measures G is equal to $\mathbf{C}$ or $\mathbf{R}$. We denote $P(\mathsf{R},G,R,U_*;R') := [(\mu,\rho_\mu,\eta_\mu) : (\mu,\rho_\mu) \in \mathsf{L}(\mathsf{R},G,R;R'),\mu$ is pseudo-differentiable and $\eta_\mu(t,U_*,A) \in L^1(\mathbf{K},v,\mathbf{C})]$, where

$$\eta_\mu(t,U_*,A) = j(t)g(t,0,b)[\mu^h(U_t^{-1}(A) - \mu^h(A)],$$

$j(t) = 1$ for each $t \in \mathbf{K}$ for $S = X$; $j(t) = 1$ for $t \in B(\mathbf{K},0,1)$, $j(t) = 0$ for $|t|_\mathbf{K} > 1$ for a topological group S that is not a Banach space X over $\mathbf{K}$, v is the Haar measure on $\mathbf{K}$

with values in $[0,\infty)$, $(U_t : t \in B(\mathbf{K},0,1))$ is an arbitrary one-parameter subgroup. On these spaces L (or P) the additional conditions are imposed:

(a) for each neighborhood (implying that it is open) $U \ni 0 \in G$ there exists a neighborhood $S \supset V \ni e$ and a compact subset V_U, $e \in V_U \subset V$, with $\mu(B) \in U$ (or in addition $\tilde{D}^b_{U_*}\mu(B) \in U$) for each B, $\mathsf{R} \ni B \in Bf(S \setminus V_U)$;

(b) for a given U and a neighborhood $R \supset D \ni 0$ there exists a neighborhood W, $S' \supset W \ni e$, (pseudo)metric $d \in P$" and $c > 0$ such that $\rho_\mu(g,x) - \rho_\mu(h,x') \in D$ (or $\tilde{D}^b_{U_*}(\mu^g - \mu^h)(A) \in U$ for $A \in Bf(V_U)$ in addition for P) whilst $g,h \in W$, $x,x' \in V_U$, $d(x,x') < c$, where (a,b) is satisfied for all $(\mu,\rho_\mu) \in \mathsf{L}$ (or $(\mu,\rho_\mu,\eta_\mu) \in P$) equicontinuously in (a) on $V \ni U_t, U_t^{-1}$ and in (b) on W and on each V_U for $\rho_\mu(g,x) - \rho_\mu(h,x')$ and $\tilde{D}^b_{U_*}(\mu^g - \mu^h)(A)$.

These conditions are justified, since for a Gaussian measure ν on a Hilbert space Z over $\mathbf{C}$ or $\mathbf{R}$ (or measures given above for a non-Archimedean Banach space Z over $\mathbf{K}$) there exists a subspace Z' dense in Z such that ν is quasi-invariant relative to Z'. Moreover, in view of Theorem 26.2[Sko74] (or Theorems 3.20, 3.25, 4.3 and 4.7 in the non-Archimedean case respectively) there exists a subspace Z" dense in Z' such that for each $\varepsilon > 0$ and each $\infty > R > 0$ there are $r > 0$ and $\delta > 0$ with $|\rho_\nu(g,x) - \rho_\nu(h,y)| < \varepsilon$ for each $\|g-h\|_{Z"} + \|x-y\|_Z < \delta$, $g,h \in B(Z",0,r)$, $x,y \in B(Z,0,R)$, where Z" is the Banach space over $\mathbf{C}$ or $\mathbf{R}$ or $\mathbf{K}$ respectively. For a group of diffeomorphisms of a Hilbert manifold over $\mathbf{R}$ or over a non-Archimedean Banach manifold we have an analogous continuity of ρ_μ for a subgroup G" of the entire group G (see [Lud96, Lud99t, Lud00a, Lud02b]).

By M_o we denote a subspace in M, satisfying (a). Henceforth, we imply that R' contains all closed subsets from S belonging to R, where G and R are complete.

For $\mu : Bf(S) \to G$ by $L(S,\mu,G)$ we denote the completion of a space of continuous $f : S \to G$ such that $\|f\|_d := \sup_{h \in C_b(S,G)} d(\int_S f(x)h(x)\,\mu(dx),0) < \infty$ for each $d \in P$", where $C_b(S,G)$ is a space of continuous bounded functions $h : S \to G$. We suppose that for each sequence $(f_n : n) \subset L(S,\mu,G)$ for which $g \in L(S,\mu,G)$ exists with $d(f_n(x),0) \le d(g(x),0)$ for every $d \in P$", x and n, that f_n converges uniformly on each compact subset $V \subset S$ with $|\mu|(V) > 0$. In the cases $G = \mathbf{C}$ it coincides with $L^1(S,\mu,\mathbf{C})$ correspondingly, hence this supposition is the Lebesgue theorem. By $Y(v)$ we denote $L^1(\mathbf{K},v,\mathbf{C})$.

Now we may define topologies and uniformities with the help of corresponding bases (see below) on $\mathsf{L} \subset G^{\mathsf{R}} \times R_o^{S'\times S} =: Y$ (or $P \subset G^{\mathsf{R}} \times R_o^{S'\times S} \times G^{S'\times K\times\mathsf{R}} =: Y$), $R_o \subset R \setminus \{0\}$. There are the natural projections $\pi : \mathsf{L}\ (or \mathsf{P}) \to M_o$, $\pi(\mu,\rho_\mu(*,*)\ (\vee\ ,\eta_\mu)) = \mu$, $\xi : \mathsf{L}\ (\vee\ P) \to R^{S'\times S}$, $\xi(\mu,\rho_\mu,(\vee\ \eta_\mu)) = \rho_\mu$, $\zeta : P \to G^{S'\times K\times\mathsf{R}}$, $\zeta(\mu,\rho_\mu,\eta_\mu) = \eta_\mu$. Let H be a filter on L or P, $U = U' \times U$" or $U = U' \times U" \times U"'$, U' and U" be elements of uniformities on G, R and $Y(v)$ correspondingly, $\tau' \ni W \ni e$, $\tau \ni V \supset V_{U'} \ni e$, $V_{U'}$ is compact. By $[\mu]$ we denote (μ,ρ_μ) for L or (μ,ρ_μ,η_μ) for P, $\Omega := \mathsf{L} \vee P$, $[\mu](A,W,V) := [\mu^g(A), \rho_\mu(g,x), \vee\ \eta_{\mu^g}(t,U_*,A)|\ g \in W, x \in V, \vee\ t \in K]$. We consider $\mathsf{A} \subset \mathsf{R}$, then

$$\mathsf{W}(\mathsf{A},W,V_{U'};U) := \{([\mu],[\nu]) \in \Omega^2 | ([\mu],[\nu])(A,W,V_{U'}) \subset U\}; \qquad (1)$$

$$\mathsf{W}(\mathsf{S};U) := \{([\mu],[\nu]) \in \Omega^2 | \{(B,g,x) : ([\mu],[\nu])(B,g,x)) \in U\} \in \mathsf{S}\}, \qquad (2)$$

where S is a filter on $\mathsf{R} \times S' \times S^c$, S^c is a family of compact subsets $V' \ni e$.

$$\mathsf{W}(\mathsf{F},W,V;U) := \{([\mu],[\nu]) \in \Omega^2 | \{B : ([\mu],[\nu])(B,g,x) \in U, g \in W, x \in V\} \in \mathsf{F}\}, \qquad (3)$$

where F is a filter on R (compare with § 2.1 and 4.1[Con84]);

$$\mathsf{W}(\mathsf{A},\mathsf{G};U) := \{([\mu],[\nu]) \in \Omega^2 | \ \{(g,x) : ([\mu],[\nu])(B,g,x) \in U, B \in \mathsf{A}\} \in \mathsf{G}\}, \qquad (4)$$

where G is a filter on $S' \times S^c$; suppose $\mathsf{U} \subset \mathsf{R} \times \tau'_e \times S^c$, Φ is a family of filters on $\mathsf{R} \times S' \times S^c$ or $\mathsf{R} \times S' \times S^c \times Y(v)$ (generated by products of filters $\Phi_{\mathsf{R}} \times \Phi_{S'} \times \Phi_{S^c}$ on the corresponding spaces), U' be a uniformity on (G,R) or $(G,R,Y(v))$, $\mathsf{F} \subset Y$. A family of finite intersections of sets $\mathsf{W}(A,U) \cap (\mathsf{F} \times \mathsf{F})$ (see (1)), where $(A,U) \in \mathsf{U} \times \mathsf{U}'$ (or $\mathsf{W}(F,U) \cap (\mathsf{F} \times \mathsf{F})$ (see (2)), where $(F,U) \in (\Phi \times \mathsf{U}')$ generate by the definition a base of uniformity of U-convergence (Φ-convergence respectively) on F and generate the corresponding topologies. For these uniformities are used notations

(i) F_{U} and F_{Φ}; $\quad \mathsf{F}_{\mathsf{R} \times W \times V}$ is for F with the uniformity of uniform convergence

on $\mathsf{R} \times W \times V$, where $W \in \tau'_e$, $V \in S^c$, analogously for the entire space Y;

(ii) F_A denotes the uniformity (or topology) of pointwise convergence for

$A \subset \mathsf{R} \times \tau'_e \times S^c =: Z$, for $A = Z$ we omit the index (see formula (1)). Henceforward, we use H' instead of H in 4.1.24[Con84], that is, $\mathsf{H}'(A,\tilde{\mathsf{R}})$-filter on R generated by the base $[(L \in \mathsf{R} : L \subset A \setminus K') : K' \in \tilde{\mathsf{R}}, K' \subset A]$, where $\tilde{\mathsf{R}} \subset \mathsf{R}$ and $\tilde{\mathsf{R}}$ is closed relative to the finite unions.

For example, let S be a locally $\mathbf{K}$-convex space, S' be a dense subspace, G be a locally $\mathbf{L}$-convex space, where $\mathbf{K},\mathbf{L}$ are fields, $R = B(G)$ be a space of bounded linear operators on G, $R_o = GL(G)$ be a multiplicative group of invertible linear operators. Then others possibilities are: $S = X$ be a Banach space over $\mathbf{K}$, $S' = J_\mu$, $S" = J^b_\mu$ as above; $S = G(t)$, $S' \supset S"$ are dense subgroups, $G = R$ be the field $\mathbf{R}$, M be an analytic Banach manifold over $\mathbf{K} \supset \mathbf{Q_p}$ (see [Lud96]). The rest of the necessary standard definitions are recalled further when they are used.

5.2. Lemma. *Let* R *be a quasi-*δ*-ring with the weakest uniformity in which each* $\mu \in \mathsf{M}$ *is uniformly continuous and* $\Phi \subset \hat{\Phi}_C(\mathsf{R}, S' \times S^c)$*. Then* $\mathsf{L}(\mathsf{R},G,R,R')_\Phi$ *(or* $P(\mathsf{R},G,R,U_*;R')_\Phi$*) is a topological space on which* R_o *acts continuously from the right.*

Proof. We recall that $\hat{\Phi} := \hat{\Phi}(X)$ denotes a family of filters F on X such that for each mapping $f : \Phi \to \mathsf{B}(X)$ with $f(\Sigma) \subset \Sigma$ for each $\Sigma \in \Phi$ there exists a finite subset $\Psi \subset \Phi$ such that $\bigcup_{\Sigma \in \Psi} f(\Sigma) \in F$, where $\mathsf{B}(X)$ denotes a family of all subsets in X, $\Phi_C(X) =: \Phi_C$ be a family of Cauchy filters on X. In view of proposition 4.2.2[Con84] the space of measures $\mathsf{M}(\mathsf{R},G;R')_\Phi$ is a topological left-R-module. On the other hand, $\rho_{\lambda\mu}(a,x) = \lambda\rho_\mu(a,x)\lambda^{-1}$ for $\lambda \in R_o$, from the continuity of the inversion $\lambda \mapsto \lambda^{-1}$ in R_o and the multiplication in R for each entourage of the diagonal $U"$ in R there exists an entourage of a diagonal $U"_1$ in R and an open neighborhood $U_4 \ni \lambda$ in R_o such that $\beta U"_1\beta^{-1} \subset U"$ for each $\beta \in U_4$. Choosing $\Psi \subset \Phi$ for a given F we find $U"_1(\Sigma)$ and $U_4(\Sigma)$ for each $\Sigma \in \Psi$, then $\bigcap_{\Sigma \in \Psi} U"_1(\Sigma) =: \tilde{U}"$ and $\tilde{U}_4 = \bigcap_{\Sigma \in \Psi} U_4(\Sigma)$ are entourages of the diagonal and generate a neighborhood of λ. This shows the continuity by λ for L and P, since for P is satisfied: $PD(b,\lambda f(x)) = \lambda PD(b, f(x))$ due to definition 4.1 for pseudo-differentiable f.

5.3. Proposition. (1). *Let* T *be a* $\hat{\Phi}_4$*-filter on* $\mathsf{M}_o(\mathsf{R},G;R')$*,* $\{A_n\}$ *be a disjoint* $\Theta(\mathsf{R})$*-sequence,* Σ *be the elementary filter on* R *generated by* $\{A_n : n \in \mathbf{N}\}$ *and* $\phi : \mathsf{M}_o \times \mathsf{R} \to G$ *with* $\phi(\mu,A) = \mu(A)$*. Then* $\phi(\mathsf{T} \times \Sigma)$ *converges to* 0.

(2). *Moreover, let* U *be a base of neighborhoods of* $e \in S'$, $\phi : \mathsf{L} \to G \times R$, $\phi(\mu, A, g) := (\mu^g(A), \rho_\mu(g,x))$, *where* $x \in A$. *Then* $(0,1) \in \lim \phi(\mathsf{T} \times \Sigma \times \mathsf{U})$.

(3). *If* T *is a* Φ_4*-filter on* $P(\mathsf{R}, G, R, U_*; R')$,
$\psi(\mu, B, g, t, U_*) = [\mu(B); \rho_\mu(g,x); \eta_{\mu^g}(t, U_*, B)]$,
then $(0,1,0) \in \lim \psi(\mathsf{T} \times \Sigma \times \mathsf{U})$ *for each given* $U_* \in S$”*, where* Σ *and* U *as in* $(1,2)$.

Proof. We recall that for a set X, Θ-net in X is a net (I, f) in X such that for each increasing sequence $(i_n : n \in \mathbf{N}) \subset I$ is satisfied $(f(i_n) : n) \in \Theta$. A sequence $(x_n : n \in \mathbf{N}) \subset X$ is called a Θ-sequence, if $f : \mathbf{N} \to X$, $f(n) = x_n$, (N, f) is a Θ-net. Then $\Phi(\Theta, X) =: \Phi(\Theta)$ denotes a family of filters on X of the form $f(F)$, where (I, f) is a Θ-net in X, F is a filter of sections $[(\lambda \in I : \lambda \geq i) : i \in I]$ for a directed set I, $\hat{\Phi}(\Theta) := \Phi(\Theta)\hat{}$. For a topological space X, $\Theta_j(X)$ denotes a family of sequences having converging subsequences for $j = 1$ or having limit points whilst $j = 2$. For a uniform space X, $\Theta_3(X)$ denotes a family of sequences having Cauchy subsequences $\Theta_4 := \Theta_2 \cup \Theta_3$, $\Phi_j(X) := \Phi(\Theta_j(X))$.

Then (1) follows from 4.2.6[Con84]. (2) From the restrictions (a,b) in 5.1 it follows that $\mu^g(A) = \int_A \rho_\mu(g,x)\mu(dx)$. If in addition $\lim \mu(\mathsf{F}(A; R')) = \mu(A)$, then $\lim \mu(\mathsf{F}(X \setminus A, R')) = \mu(X \setminus A)$, since by the definition 4.1.24[Con84] for each $\tau_G \ni D \ni 0$ for $A \in \mathsf{R}$ there exists $\tau_S \ni V \supset A$ with $\mu(V) - \mu(A) \in D$, where $\mathsf{F}(A, R')$ is a filter generated by A and R'. Due to the Radonian property of μ and μ^g we get $\lim_{\mathsf{T} \times \Sigma} \mu^g(A) = 0$.

(3). Additionally to (1,2) it remains to verify that η_μ converges. In view of Theorems 4.4 and 4.7 (or §5.1) a pseudo-differential of order $b : \tilde{D}^b_{U_*}\mu = \nu$ is a measure for pseudo-differentiable μ. Due to (1) and the $\hat{\Phi}_4$-condition we get $(0,1,0) \in \lim \psi(\mathsf{T} \times \Sigma \times \mathsf{U})$ for a given U_*, since in §§ 4.1 and 5.1 the integral is by $t \in \mathbf{K}$ for the Banach space X over $\mathbf{K}$ and by $B(\mathbf{K}, 0, 1)$ correspondingly for S that is not the Banach space.

5.4. Proposition. *Let* H *be a* $\hat{\Phi}_4$*-filter on* L *(or* P*) with the topology* F *(see* § 5.1.(ii)*),* $A \in \mathsf{R}$, $\tau_G \ni U \ni 0$, $\mathsf{H}'(A, R') \in \Psi_f(\mathsf{R})$. *Then there are* $L \in \mathsf{H}$, $\tilde{K} \in R'$ *and an element of the uniformity* U *for* $\mathsf{L}_{R'}$ *or* $P_{R'}$ *such that* $\tilde{\mathbf{K}} \subset A$, $L = [(\mu, \rho_\mu(g,x)) : M := \pi_{\mathsf{M}_\circ}(L) \ni \mu, \pi_{\tau'_e}(L) =: W \ni g$ *(or* $(\mu, \rho_\mu, \eta_\mu(*, *, U_*))$ *and additionally* $\tilde{D}^b_{U_*}\mu = PD(b, \eta_\mu)$*)*$]$, $e \in W \in \tau'$, $\mu^g(B) - \nu^h(C) \in U$ *(or in addition* $(\tilde{D}^b_{U_*}\mu^g(B)) - (\tilde{D}^b_{U_*}\nu^h(C)) \in U$*) for* $\tilde{K} \subset B \subset A$, $\tilde{K} \subset C \subset A$ *for each* $([\mu], [\nu]) \in \overline{\overline{L}}^2 \cap \mathsf{U}$, *where* $\overline{\overline{L}} := cl(L, \mathsf{L}_{R'})$ *(or* $cl(L, P_{R'})$*),* $\pi_{\mathsf{M}_\circ}$ *is a projector from* L *into* $\mathsf{M}_\circ$.

Proof. We recall that $\Psi(\mathsf{R}) := \hat{\Phi}(\Theta(\mathsf{R}))$, where $\Theta(\mathsf{R})$ is a family of sequences $(A_n : n \in \mathbf{N}) \subset \mathsf{R}$ for which there exists $\Omega \in \Sigma(\mathsf{R})$ with $(n \in \mathbf{N} : A_n \in \Omega)$ is infinite, $\Sigma(\mathsf{R}) := [(\bigcup_{n \in J} A_n : J \in \mathsf{B}(\mathbf{N})) : (A_n : n \in \mathbf{N}) \in \Gamma(\mathsf{R})]$, $\Gamma(\mathsf{R})$ is a family of disjoint sequences $(A_n : n \in \mathbf{N}) \subset \mathsf{R}$ for which $[\bigcup_{n \in J} A_n : J \subset \mathbf{N}] \subset \mathsf{R}]$. A ring of sets Z is called a quasi-δ-ring, if each disjoint sequence $(A_n : n \in \mathbf{N})$ from R the union of which $\bigcup_n A_n = A$ is contained in a set $B \in \mathsf{R}$, $A \subset B$, has a subsequence $(A_{n_j} : j \in \mathbf{N}) \in \Gamma(\mathsf{R})$.

From Proposition 4.2.7[Con84] and the Radonian property of measures we have $\mu(B) - \nu(C) \in U'$ for each $(\mu, \nu) \in \overline{\overline{(\mathsf{M} \times \mathsf{M})}} \cap \pi_{\mathsf{M}_0}(\mathsf{U})$. Then for each element D' of the uniformity on R there are $d \in P$”, $c > 0$ and L such that $\rho_\mu(g,x) - \rho_\nu(h,x') \in D'$ for $g, g' \in W$, $\mu, \nu \in M$, $d(x,x') < c$, since $\mathsf{H} \in \hat{\Phi}_4$, where $\overline{\overline{\mathsf{M}}} := \mathsf{cl}(\mathsf{M}, \mathsf{E}(\mathsf{R}, \mathsf{G}; \mathsf{R}')_{\mathsf{R}'})$ is the closure of $\mathsf{M} \subset E(\mathsf{R}, G; R')$ in $E(\mathsf{R}, G; R')_{R'}$, $E(\mathsf{R}, G; R') := [\mu \in E(\mathsf{R}, G) : \mu \text{ is } R'\text{-regular}]$, that is, $\mu(F(A, R'))$ converges to $\mu(A)$ for each $A \in R$ (simply regular, if R' consists of closed subsets and conditions in definition 11.34[HR79] are satisfied). Then $F(A, R')$ is a filter on R generated by a base $[(B \in \mathsf{R} : \tilde{K} \subset B \subset A) : \tilde{K} \subset A, \tilde{K} \in R']$, $E(\mathsf{R}, G)$ is a set of exhaustive additive maps μ, that is, $\mu(A_n)$ converges to 0 for each $(A_n : n \in \mathbf{N}) \in \Gamma(\mathsf{R})$.

Then $\mu^g(B) - \mu(B) \in U'$, $\nu^h(C) - \nu(C) \in U'$ and for $3U' \subset U$ we get 5.4 for L. From Theorems 4.4 and 4.7 (or § 5.1), the Egorov conditions and the Lebesgue theorem we get 5.4 for P, since μ are probability measures and $\mathsf{L}_{R'}$ (or $P_{R'}$) correspond to uniformity from § 5.1.(ii) with $A = R' \times \tau'_e \times S^c$. Indeed, $\mu^g(A) - \nu^h(A) = (\mu^g(A) - \mu^g(V_{U'})) + (\mu^g(V_{U'}) - \nu^h(V_{U'})) + (\nu^h(V_{U'}) - \nu^h(A))$, $\mu^g(A) = \int_A \rho_\mu(g,x)\mu(dx)$ for each $A \in Bf(S)$, for each $\tau_G \ni U' \ni 0$ there exists a compact subset $V'_U \subset A$ with $\mu^g(B) \in U'$ for each $B \in Bf(A \setminus V_{U'})$ and the same for ν^h (due to the Condition in 5.1 that R' contains $Bf(S)$). From the separability of S, S' and the equality of their topological weights to $\aleph_0$, restrictions 5.1(a,b) it follows that there exists a sequence of partitions $Z_n = [(x_m, A_m) : m, x_m \in A_m]$ for each $A \in Bf(S)$, $A_i \cap A_j = \emptyset$ for each $i \neq j$, $\bigcup_m A_m = A$, $A_m \in Bf(S)$, such that $\lim_{n\to\infty}(\mu^g(A) - \sum_j \rho_\mu(g,x_j)\mu(A_j)) = 0$ and the same for ν, moreover, for $V_{U'}$ each Z_n may be chosen finite. Then there exists $W \in \tau'_e$ with $W \times (S \setminus V^2) \subset (S \setminus V)$, $\tau_e \ni V \subset V^2$, $\nu^g(B)$ and $\mu^g(B) \in U'$ for each $B \in Bf(S \setminus V^2)$ (for $G = \mathbf{R}$) and $g \in W$ (see 5.1.(a)). Then from $A = [A \cap (S \setminus V^2)] \cup [A \cap V^2]$ and the existence of compact $V'_{U'} \subset V$ with $\mu(E) \in U'$ for each $E \in Bf(V \setminus V'_{U'})$ and the same for ν, moreover, $(V'_{U'})^2$ is also compact, it follows that $\mu^g(B) - \nu^h(C) \in U$ for $9U' \subset U$, since $R' \supset Bf(S)$, where W satisfies the following condition $\mu^g(V'_{U'}) - \nu^h(V'_{U'}) \in U'$ for $V'_{U'} \subset V^2$ due to 5.1.(b), $\mu(B) - \nu(C) \in U'$, $WV'_{U'} \subset (V'_{U'})^2$ due to precompactness of W in S. Since pseudo-differentiable measures are also quasi-invariant, hence for them Proposition 5.4 is true.

Now let $[\mu] \in \lim H$, $A \in Bf(S)$, then $\eta_\mu \in \lim \zeta(H)$ in $Y(\nu)$ and there exists a sequence η_{μ_n} such that $\int_{\mathbf{K}} \eta_{\mu_n}(\lambda, U_*, A)\nu(d\lambda) = \tilde{D}^b_{U_*}\mu_n(A)$ due to §§ 4.4, 4.7 or 5.1 and $\lim_{n\to\infty} \tilde{D}^b_{U_*}\mu_n(A) = \int_{\mathbf{K}} \eta_\mu(\lambda, U_*, A)\nu(d\lambda) =: \kappa(A)$ due to the Lebesgue theorem. From $\eta_\mu(\lambda, U_*, A \cup B) = \eta(\lambda, U_*, A) + \eta(\lambda, U_*, B)$ for $A \cap B = \emptyset$, $B \in Bf(S)$ and the Nikodym theorem [Con84] it follows that $\nu(A)$ is the measure on $Bf(S)$, moreover, $\kappa(A) = \tilde{D}^b_{U_*}\mu(A)$. Since $\mu^g(A) = \int_A \rho_\mu(g,x)\mu(dx)$ for $A \in Bf(S)$ for $g \in S'$, then

$$\eta_{\mu^g}(\lambda, U_*, A) = j(\lambda)g(\lambda, 0, b)[\mu^g(A) - \mu^g(U_\lambda^{-1}A)]$$

$$= j(\lambda)g(\lambda, 0, b)\int_A \rho_\mu(g,x)[\mu(dx) - \mu^{U_\lambda}(dx)]$$

and in view of the Fubini theorem there exists

$$\tilde{D}^b_{U_*}\mu^g(A) = \int_A \left[\int_K \rho_\mu(g,x)j(\lambda)g(\lambda,0,b)[\mu(dx) - \mu^{U_\lambda}(dx)\right]\nu(d\lambda),$$

where $j(t) = 1$ for $S = X$ and $j(t)$ is the characteristic function of $B(\mathbf{K}, 0, 1)$ for S that is not the Banach space X. Then μ-a.e. $\tilde{D}^b_{U_*}\mu^g(dx)/\tilde{D}^b_{U_*}\mu(dx)$ coincides with $\rho_\mu(g,x)$ due to 5.1.(a,b), hence, $(\tilde{D}^b_{U_*}\mu^g, \rho_{\mu^g})$ generate the Φ_4-filter in L arising from the $\hat{\Phi}_4$-filter in P. Then we estimate $\tilde{D}^b_{U_*}(\mu^g - \nu^h)(A)$ as above $\mu^g(A) - \nu^h(A)$. Therefore, we find for the Φ_4-filter corresponding L, since there exists $\delta > 0$ such that $U_\lambda \in W$ for each $|\lambda| < \delta$. For Φ_4-filter we use the corresponding finite intersections $W_1 \cap \cdots \cap W_n = W$, where W_j correspond to the Φ_4-filters H_j.

5.5. Corollary. *If* $\{\mathsf{H}'(A, R') : A \in \mathsf{R}\} \subset \Psi_f(\mathsf{R})$, T *is a* $\hat{\Phi}_4$*-filter in* L, U *is an element of uniformity in* $\underline{\mathsf{L}}$ *(or* P*), then there are* $L \in \mathsf{T}$ *and an element* V *of the uniformity in* $\mathsf{L}_{R'}$ *(or* $P_{R'}$*) such that* $\bar{\bar{L}}^2 \cap \mathsf{V} \subset \bar{\bar{L}}^2 \cap \mathsf{U}$.

Proof. In view of Corollary 4.2.8[Con84] and Theorem 5.4 above for $\mathsf{U_M} = \pi_\mathsf{M} \times \pi_\mathsf{M} \mathsf{U}$ there exists an entourage of the diagonal (see about uniform spaces in [Eng86]), V_M in $E(\mathsf{R},G;R')_{R'}$, such that $\bar{\bar{M}}^2 \cap V_\mathsf{M} \subset \bar{\bar{M}}^2 \cap \mathsf{U_M}$, where $\pi_\mathsf{M} : \mathsf{L} \to \mathsf{M}$ (or $P \to \mathsf{M}$) is a projector, $\mathsf{M}(\mathsf{R},G)$ is a set of measures on R with values in G, $\mathsf{M}(\mathsf{R},G;R') := [\mu \in \mathsf{M}(\mathsf{R},G) : \mu$ is R'-regular $]$. Since $\mathsf{M}(\mathsf{R},G;R')_{R'} \subset E(\mathsf{R},G;R')_{R'}$, then there exists an entourage of the diagonal, V in $\mathsf{L}_{R'}$ (or $P_{R'}$), such that $\pi_\mathsf{M} \times \pi_\mathsf{M} V \subset V_\mathsf{M} \cap \bar{\bar{M}}^2$.

5.6. Lemma. *Let* R *be a quasi-σ-ring directed by the inclusion,* U *be an upper directed subset in* R, H *be a* $\hat{\Phi}_4$*-filter in* L (*or* P), $0 \in U \ni \tau_G$. *Then there are* $M \in \mathsf{H}$, $A \in \mathsf{U}$, *an element of the uniformity* U' *in* $\mathsf{L}_{R'}$ (*or* $P_{R'}$) *such that* $\mu^g(B) - \nu^h(C) \in U$ (*or in addition* $\tilde{D}^b_{U_*}\mu^g(B) - \tilde{D}^b_{U_*}\nu^h(C) \in U$) *for each* $([\mu],[\nu]) \in \bar{\bar{L}}^2 \cap U'$, $g,h \in W$, W *is defined by a projection of* U' *onto* S', $B,C \in \mathsf{U}$ *with* $A \le B$, $A \le C$.

Proof. For each C and $E \in \mathsf{U}$ by the definition there are $F \in \mathsf{U}$ with $C \le F$ and $E \le F$. There are open subsets P and V in $\underline{S}$ and $W \in \tau'_e$ with $WP \subset P^2 \subset V$ such that $(\mu - \nu)(B) \in U$ for each $B \subset S \setminus P$ and $(\mu,\nu) \in \bar{M}^2 \cap \pi_{\mathsf{M_o}}(U')$. Indeed, $\pi_{\mathsf{M_o}}\mathsf{H}$ is a base of a filter T in $\mathsf{M_o}$. Therefore, $g(B \cap S \setminus P^2) \subset S \setminus P$ for $g \in W$, consequently, $[\mu^g - \nu^g](B \cap S \setminus P^2) \in U$. In view of § 5.1.(a,b) for D,V there are W, $c > 0$, $d \in P$" such that $\rho_\mu(g,x) - \rho_\nu(h,x') \in D$ for $g,h \in W$ and $d(x,x') < c$, $x,x' \in V_c$, consequently, $\mu^g(B) - \nu^h(C) = [\mu^g(B \cap P^2) - \nu^h(C \cap P^2)] + [\mu^g(B \setminus P^2) - \nu^h(C \setminus P^2)] \in 3(DU + U)$. Modifying the proof of § 4.2.9 [Con84] we get the statement of this lemma for L. For P an estimate of $\tilde{D}^b_{U_*}\mu^g(B) - \tilde{D}^b_{U_*}\nu^h(C)$ may be done analogously to the proof of Proposition 5.4.

5.7. Theorem. *Let* H *be a* $\hat{\Phi}_4$*-filter in* L (*or* P), $\{A_n : n \in \mathbf{N}\} \in \Gamma(\mathsf{R})$, $\tau_G \ni U \ni 0$. *Then there are* $L \in \mathsf{H}$, $M \in \mathsf{B}(\mathbf{N})$ *and an element* U *of the uniformity in* L *such that* $\mu^g(\cup(A_n : n \in M') - \nu^h(\cup(A_n : n \in M") \in U$ (*or in addition* $(\tilde{D}^b_{U_*}\mu^g(\bigcup_{n\in M'}) - \tilde{D}^b_{U_*}\nu^h(\bigcup_{n\in M"} A_n)) \in U$ *for each* $([\mu],[\nu]) \in \bar{L}^2 \cap \mathsf{U}$ *and* $M',M" \in \mathsf{B}(\mathbf{N})$ *with* $M \cap M' = M \cap M"$. *If* $\{\mathsf{H}'(A,R')\} \subset \Psi_f(\mathsf{R})$, *then* U *may be chosen as an element of the uniformity in* $\mathsf{L}_{R'}$ (*or* $P_{R'}$), *where* $\bar{L} := cl(L,\mathsf{L})$ (*or* $\bar{L} := cl(L,P)$).

Proof. We recall that $\mathsf{B}_f(\mathbf{N})$ denotes the family of finite subsets in $\mathbf{N}$, $\Psi_f(\mathsf{R}) := \hat{\Phi}(\Theta_f(\mathsf{R}))$, $\Theta_f(\mathsf{R})$ is a family of sequences $(A_n : n \in \mathbf{N}) \subset \mathsf{R}$ for which there exists $\Omega \in \Sigma_f(\mathsf{R})$ with $card[n \in \mathbf{N} : A_n \in \Omega] = \aleph_0$, $\Sigma_f(\mathsf{R}) := [(\bigcup_{n\in M} A_n : M \in \mathsf{B}_f(\mathbf{N})) : (A_n : n \in \mathbf{N}) \in \Gamma(\mathsf{R})]$. Let $\phi(M) := \cup\{A_n : n \in M\}$, $M \subset \mathsf{B}_f(\mathbf{N})$, Σ be a filter of neighborhoods of $\emptyset$ in $\mathsf{B}_f(\mathbf{N})$, then $\phi(\Sigma) \in \Psi_f(\mathsf{R})$. In view of § 4.1.14 [Con84] we have $\mu(\phi(\Sigma)) \to 0$ for each $\mu \in \mathsf{M}(\mathsf{R},G,R')$. We take a symmetric neighborhood $V' \ni 0$ in G with $3V' \subset U$. Then there are $L \in \mathsf{H}$ and $M \in \mathsf{B}_f(\mathbf{N})$ such that $\mu^g(\phi(M')) - \nu^h(\phi(M")) \in V'$ (or in addition $\tilde{D}^b_{U_*}\mu^g(\phi(M')) - \tilde{D}^b_{U_*}\nu^h(\phi(M")) \in V'$) for each $([\mu],[\nu]) \in \bar{L}^2 \cap \mathsf{U}$ and $M',M" \in \mathsf{B}(\mathbf{N} \setminus M)$ (see Lemma 5.6). We choose U such that $\mu^g(\phi(M_o)) - \nu^h(\phi(M_o)) \in V'$ (or in addition $\tilde{D}^b_{U_*}\mu^g(\phi(M_o)) - \tilde{D}^b_{U_*}\nu^h(\phi(M_o)) \in V'$) for each $([\mu],[\nu]) \in \mathsf{U}$ and $M_o \subset M$, consequently, $\mu^g(\phi(M')) - \nu^h(\phi(M")) \in 3V' \subset U$ (or additionally $\tilde{D}^b_{U_*}\mu^g(\phi(M')) - \tilde{D}^b_{U_*}\nu^h(\phi(M")) \in 3V'$) for $M \cap M' = M \cap M"$, the last statement follows from Corollary 5.5.

5.8. Corollary. *Let* $(A_n : n \in \mathbf{N}) \in \blacksquare(\mathsf{R})$, H *be a* $\hat{\Phi}_4$*-filter in* L *(or P)*, $\tau_G \ni U \ni 0$, *then there are* $(L,M) \in \mathsf{H} \times \mathsf{B}_f(\mathbf{N})$ *such that* $\mu^g(\cup(A_n : n \in M')) \in U$ (*or additionally* $\tilde{D}^b_{U_*}\mu^g(\bigcup_{n\in M'} A_n) \in U$) *for* $[\mu] \in \bar{L}$ *and* $M' \in \mathsf{B}(\mathbf{N} \setminus M)$.

Proof. Let us take a fixed ν and W from § 5.7 with $\nu(\bigcup_{n\in M'} A_n) \in \tilde{U}$ (or additionally $\tilde{D}^b_{U_*}\nu(\bigcup_{n\in M'} A_n) \in \tilde{U}$) and $(\mu^g - \nu)(\bigcup_{n\in M'} A_n) \in \tilde{U}$), where $2\tilde{U} \subset U$, $0 \in \tilde{U} \in \tau_G$, $g \in W$.

5.9. Theorem. *Let* $\mathsf{U} \subset \mathsf{R} \times \tau'_e \times S^c =: Z$ *be such that* $id : \mathsf{L_U} \to \mathsf{L}$ (*or* $id : P_\mathsf{U} \to P$) *be* *uniformly* Φ_4*-continuous,* $\{\mathsf{H}'(A,R') : A \in \mathsf{R}\} \subset \Psi_f(\mathsf{R})$*. Then*

(a) L *(or P) is* Φ_4*-closed in* $G^{\mathsf{R}} \times R_o^{S'\times S}$ *(or* $G^{\mathsf{R}} \times R_o^{S'\times S} \times G^{S'\times K\times \mathsf{R}}$*),* $\mathsf{M}_\circ$ *is* Φ_4*-closed in* G^{R}*;*

(b) *if G and R are* Φ_i*-compact, then* L_{U} *(or* P_{U}*) is* Φ_i*-compact, where* $i \in (1,2,3,4)$*;*

(c) *if* $G \times R_o$ *is sequentially complete, then* L_{U} *(or* P_{U}*) is also;*

(d) *if* $(0,0)$ *is the* G_δ*-subset in* $G \times R$*, then* L_{U} *(or* P_{U}*) is* Φ_2*-compact; if additionally* $G \times R$ *is sequentially complete, then* L_{U} *(or* P_{U}*) is* Φ_4*-compact;*

(e) L_{U} *(or* P_{U}*) is Hausdorff.*

Proof. We recall that a subset $A \subset E$ of a topological or of a uniform space E is called Φ_i-closed (or compact), if for each $\Phi_i(A)$-filter F is satisfied $\lim F \subset A$ (or $\lim F \neq \emptyset$ respectively, that is, this definition of compactness differs from the usual). In view of Theorem 4.2.14[Con84] and Proposition 5.4 is accomplished 5.9(a).

From §§ 2.1.14, 1.8.11[Con84] and Theorem 5.7, Φ_i-compactness of M and $\mathsf{M}_\circ$, also from the completeness of R_o, $L^1(K,\nu,\mathbf{C})$ it follows (b).

Then (c) follows from (a) and § 1.8.7[Con84];

(d) follows from (c) and § 1.6.4[Con84], since $G \times R$ is Φ_2-compact. From $\mathsf{M}(\mathsf{R},G;R')_{\mathsf{U}}$ (see §4.2.14[Con84]), $R_o^{S'\times S}$ and $L^1(K,\nu,\mathbf{C})$ being Hausdorff it follows (d).

5.10. Theorem. *Let* Φ *be a set of* $\hat{\Phi}_4$*-filters on* L *(or P), also let* $\phi : \mathsf{L} \times \mathsf{R} \times S' \times S(\vee \times \mathbf{K}) \to G \times R_o(\vee \times Y(\nu))$ *be such that* $(\phi(\mu,\rho_\mu(*,*)(\vee\, \eta_\mu),A,g,x) := (\mu(A),\rho_\mu(g,x)(\vee\, \eta_{\mu^g}(t,U_*,A)))$*. If* $\{\mathsf{H}'(A,R')|A \in \mathsf{R}\} \subset \Psi_f(\mathsf{R})$*, then a mapping* $\phi : \mathsf{R} \times S' \times S(\vee \times \mathbf{K}) \to G_{\Phi'}^{\mathsf{M}_\circ(\mathsf{R}),G;R'} \times R_o^{S'\times S}(\vee\ \times Y(\nu))$ *gives* R'*-regular quasi-invariant measure (or in addition a pseudo-differentiable measure)* $(A,g,x) \to \phi(*,A,g,x)$ *(or* $(A,g,x,t) \to \phi(*,A,g,x,t)$*), satisfying Conditions* $5.1.(a,b)$*, where* $\Phi' := \pi_{\mathsf{M}_\circ}(\Phi)$*, in* $R_o^{S'\times S}$ *the topology corresponds to the topology in* L*.*

Proof follows from §§ 5.4 and 5.8.

5.11. Note. Let it be a sequence $\{[\mu_n] : n \in \mathbf{N}\}$ of quasi-invariant measures (or pseudo-differentiable measures) converging uniformly in uniformity 5.1.(ii) and fulfilling Conditions 5.1.(a,b) then in accordance with Corollary 5.8 $\{(\mu_n)^g : n \in \mathbf{N}\}$ (or also $[\tilde{D}^b_{U_*}\mu^g_n : n]$) is uniformly σ-additive for each fixed $g \in S'$. Moreover, it is uniformly σ-additive by $g \in W$ for each given $B \in \mathsf{R}$ such that $gB \subset V$ for suitable open W in S' and V in S. For L this means that for each $0 \in D \in \tau_R$ and $e \in U \in \tau_G$ there are $W \ni e$, $d \in P$”, $c > 0$, n and $V \ni e$, a compact subset V_U, $e \in V_U \subset V$ with $\mu_m(C) \in U$ (or $\tilde{D}^b_{U_*}\mu_m(C) \in U$ in addition for P) for $C \in \mathsf{R}$ and $C \subset S \setminus V_U$, with $\rho_{\mu_m}(g,x) - \rho_{\mu_j}(h,x') \in D$ (or in addition $\tilde{D}^b_{U_*}(\mu^g_m - \mu^h_j)(A) \in U$ for $A \in Bf(V_U)$ for P) whilst $g,h \in W$ for each $x,x' \in V_U$ with $d(x,x') < c$ and $m, j > n$. In view of Theorem 5.9 there exists $\lim_{n\to\infty}(\mu_n,\rho_{\mu_n}(g,x)) = (y,d(y;g,x) \in \mathsf{L}$, that is, a quasi-invariant measure (or pseudo-differentiable measure $\lim_n[\mu]_n = [\mu] \in P$). Therefore, they are analogs of the Radon theorem for quasi-invariant measure and pseudo-differentiable measures.

1.6. Measures with Particular Properties

1. Note. In [Lud00f, Lud99s] non-Archimedean polyhedral expansions of ultra-uniform spaces were investigated and the following theorem was proved (see also Appendix B).

2. Theorem. *Let X be a complete ultra-uniform space and $\mathbf{K}$ be a locally compact field. Then there exists an irreducible normal expansion of X into the limit of the inverse system $S=\{P_n, f^m_n, E\}$ of uniform polyhedra over $\mathbf{K}$, moreover,* $\lim S$ *is uniformly isomorphic with X, where E is an ordered set, $f^m_n : P_m \to P_n$ is a continuous mapping for each $m \ge n$; particularly for the ultra-metric space (X,d) with the ultra-metric d the inverse system S is the inverse sequence.*

3. Theorem. *Let X be a complete separable ultra-uniform space and let $\mathbf{K}$ be a locally compact field. Then for each marked $b \in \mathbf{C}$ there exists a nontrivial measure μ on X which is a restriction of a measure ν in a measure space $(Y, Bf(Y), \nu) = \lim\{(Y_m, Bf(Y_m), \nu_m), \bar{f}^m_n, E\}$ on X and ν_m is quasi-invariant and pseudo-differentiable for $b \in \mathbf{C}$ relative to a dense subspace Y'_m for each m, where $Y_n := c_0(\mathbf{K}, \alpha_n)$, $\bar{f}^m_n : Y_m \to Y_n$ is a normal (that is, $\mathbf{K}$-simplicial non-expanding) mapping for each $m \ge n \in E$, $\bar{f}^m_n|_{P_m} = f^m_n$. Moreover, if X is not locally compact, then the family $\mathcal{F}$ of all such μ contains a subfamily $\mathcal{G}$ of pairwise orthogonal measures with the cardinality $card(\mathcal{G}) = 2^{\mathsf{c}}$, where $\mathsf{c} := card(\mathbf{R})$.*

Proof. Choose a polyhedral expansion of X in accordance with Theorem 2. Each mapping f^m_n is $\mathbf{K}$-simplicial, that is, f^m_n is affine on each simplex s of a polyhedra P_m and $f^m_n(s)$ is a simplex in P_n, also each f^m_n is non-expanding: $\rho_{P_n}(f^m_n(x), f^m_n(y)) \le \rho_{P_m}(x,y)$ for each $x, y \in P_m$. Since Y_n is totally disconnected normed space and each simplex in P_m is the corresponding ball in Y_m and each P_m is the uniform polyhedra, that is,

(i) $\sup\{diam(s_i : i)\} < \infty$, and

(ii) $\inf\{dist(s_i, s_j) : i \ne j\} > 0$,

then f^m_n can be extended to a normal mapping $\bar{f}^m_n : Y^m \to Y_n$ and such that Y_m can be supplied with the corresponding uniform polyhedral structure (that is, partition into disjoint union of simplices satisfying Conditions (i, ii) above). Since X is separable and $\mathbf{K}$ is a locally compact field, then each space Y_n is of countable type over $\mathbf{K}$ and E can be chosen countable. On each X_n take a probability measure ν_n such that $\nu_n(X_n \setminus P_n) < \varepsilon_n$, $\sum_{n\in E} \varepsilon_n < 1/5$. In accordance with § 3.20.1 and § 4.2.1 each ν_n can be chosen quasi-invariant and pseudo-differentiable for $b \in \mathbf{C}$ relative to a dense $\mathbf{K}$-linear subspace Y'_n. Since E is countable and ordered and each Y_m is supplied with the uniform polyhedral structure and the mapping $\bar{f}^m_n$ is normal for each $m \ge n$, then a family ν_n can be chosen by transfinite induction consistent, that is, $\bar{f}^m_n(\nu_m) = \nu_n$ for each $m \ge n$ in E, $\bar{f}^m_n(Y'_m) = Y'_n$. Then $X = \lim\{P_n, f^m_n, E\} \hookrightarrow Y$. Since $\bar{f}^m_n$ are $\mathbf{K}$-linear, then $(\bar{f}^m_n)^{-1}(Bf(Y_n)) \subset Bf(Y_m)$ for each $m \ge n \in E$. Therefore, ν is correctly defined on the algebra $\bigcup_{n\in E} f_n^{-1}(Bf(Y_n))$ of subsets of Y, where $f_n : X \to X_n$ are $\mathbf{K}$-linear continuous epimorphisms. Since ν is nonnegative and bounded by 1, then by the Kolmogorov theorem ν has an extension on the σ-algebra $Bf(Y)$ and hence on its completion $Af(Y,\nu)$. Put $Y' := \lim\{Y'_m, \bar{f}^m_n, E\}$. Then ν_m on Y_m is quasi-invariant and pseudo-differentiable for $b \in \mathbf{C}$ relative to Y'_m. From $\sum_n \varepsilon_n < 1/5$ it follows, that $1 \ge \mu(X) \ge \prod_n (1-\varepsilon_n) > 1/2$, hence μ is nontrivial.

To prove the latter statement use the Kakutani theorem for $\prod_n Y_n$ and then consider the embeddings $X \hookrightarrow Y \hookrightarrow \prod_n Y_n$ such that projection and subsequent restriction of the measure $\prod_n \nu_n$ on Y and X are nontrivial, which is possible due to the proof given above. If $\prod_n \nu_n$ and $\prod_n \nu'_n$ are orthogonal on $\prod_n Y_n$, then they give measures ν and ν' which are orthogonal on X.

4. Notes. In [Lud00a] and in [BV97, Eva89, Eva91, Eva93, Koc95, Koc96, Lud0341, Sat94] analogs of a Gaussian measure and of a Wiener measure were considered. That construction is generalized below and additional properties are proved concerning moments of a Gaussian measure and an analog of the Itô formula. Others constructions are discussed in comments (see §1.7).

Let X be a locally $\mathbf{K}$-convex space equal to a projective limit $\lim\{X_j,\phi^j_l,\Upsilon\}$ of Banach spaces over a local field $\mathbf{K}$ such that $X_j = c_0(\alpha_j,\mathbf{K})$, where the latter space consists of vectors $x=(x_k : k\in\alpha_j)$, $x_k\in\mathbf{K}$, $\|x\| := \sup_k |x_k|_{\mathbf{K}} < \infty$ and such that for each $\varepsilon>0$ the set $\{k : |x_k|_{\mathbf{K}} > \varepsilon\}$ is finite, α_j is a set, that is convenient to consider as an ordinal due to Kuratowski-Zorn lemma [Eng86, Roo78]; Υ is an ordered set, $\phi^j_l : X_j \to X_l$ is a $\mathbf{K}$-linear continuous mapping for each $j\ge l\in\Upsilon$, $\phi_j : X\to X_j$ is a projection on X_j, $\phi_l\circ\phi^j_l=\phi_j$ for each $j\ge l\in\Upsilon$, $\phi^l_k\circ\phi^j_l=\phi^j_k$ for each $j\ge l\ge k$ in Υ. Consider also a locally $\mathbf{R}$-convex space, that is a projective limit $Y=\lim\{l_2(\alpha_j,\mathbf{R}),\psi^j_l,\Upsilon\}$, where $l_2(\alpha_j,\mathbf{R})$ is the real Hilbert space of the topological weight $w(l_2(\alpha_j,\mathbf{R})) = card(\alpha_j)\aleph_0$. Suppose B is a symmetric non-negative definite (bilinear) nonzero functional $B : Y^2\to\mathbf{R}$.

5. Definitions and Notes. A measure $\mu=\mu_{q,B,\gamma}$ on X with values in $\mathbf{R}$ is called a q-Gaussian measure, if its characteristic functional $\hat{\mu}$ has the form

$$\hat{\mu}(z) = \exp[-B(v_q(z),v_q(z))]\chi_\gamma(z)$$

on a dense $\mathbf{K}$-linear subspace $\mathsf{D}_{q,B,X}$ in X^* of all continuous $\mathbf{K}$-linear functionals $z : X\to\mathbf{K}$ of the form $z(x) = z_j(\phi_j(x))$ for each $x\in X$ with $v_q(z)\in\mathsf{D}_{B,Y}$, where B is a nonnegative definite bilinear $\mathbf{R}$-valued symmetric functional on a dense $\mathbf{R}$-linear subspace $\mathsf{D}_{B,Y}$ in Y^*, $B : \mathsf{D}^2_{B,Y}\to\mathbf{R}$, $j\in\Upsilon$ may depend on z, $z_j : X_j\to\mathbf{K}$ is a continuous $\mathbf{K}$-linear functional such that $z_j=\sum_{k\in\alpha_j} e^k_j z_{k,j}$ is a countable convergent series such that $z_{k,j}\in\mathbf{K}$, e^k_j is a continuous $\mathbf{K}$-linear functional on X_j such that $e^k_j(e_{l,j})=\delta^k_l$ is the Kroneker delta symbol, $e_{l,j}$ is the standard orthonormal (in the non-Archimedean sense) basis in $c_0(\alpha_j,\mathbf{K})$, $v_q(z)=v_q(z_j) := \{|z_{k,j}|^{q/2}_{\mathbf{K}} : k\in\alpha_j\}$. It is supposed that z is such that $v_q(z)\in l_2(\alpha_j,\mathbf{R})$, where q is a positive constant, $\chi_\gamma(z) : X\to S^1$ is a continuous character such that $\chi_\gamma(z)=\chi(z(\gamma))$, $\gamma\in X$, $\chi : \mathbf{K}\to S^1$ is a character of $\mathbf{K}$ as an additive group (about a character see, for example, § VI.25 [HR79] and § III.1 [VVZ94]).

If Y is a Hilbert space with a scalar product $(*,*)$, then due to the Riesz theorem there exists $E\in L(Y)$ such that $B(y_1,y_2)=(Ey_1,y_2)$ for each y_1, $y_2\in Y$. A symmetric non-negative definite operator E (or sometimes the corresponding B) is called a correlation operator of a measure μ.

6. Proposition. *A q-Gaussian measure on X is σ-additive on some σ-algebra A of subsets of X. Moreover, a correlation operator B is of class L_1, that is, $Tr(B)<\infty$, if and only if each finite dimensional over* $\mathbf{K}$ *projection of μ is a σ-additive q-Gaussian Borel measure.*

Proof. From Definition 5 it follows, that each one dimensional over $\mathbf{K}$ projection $\mu_{x\mathbf{K}}$ of a measure μ is σ-additive on the Borel σ-algebra $Bf(\mathbf{K})$, where $0\ne x=e_{k,l}\in X_l$. Therefore, μ is defined and finite additive on a cylindrical algebra $\mathsf{U} := \bigcup_{k_1,\dots,k_n;l}\phi_l^{-1}[(\phi^l_{k_1,\dots,k_n})^{-1}(Bf(span_{\mathbf{K}}\{e_{k_1,l},\dots,e_{k_n,l}\}))]$, where $\phi^l_{k_1,\dots,k_n} : X_l\to span_{\mathbf{K}}(e_{k_1,l},\dots,e_{k_n,l})$ is a projection on a $\mathbf{K}$-linear span of vectors $e_{k_1,l},\dots,e_{k_n,l}$. This means that μ is a bounded quasi-measure on U. Since $\hat{\mu}$ is the positive definite function, then μ is real-valued. In view of the non-Archimedean analog of the Bochner-

Kolmogorov theorem (see § 2.27 above) μ has an extension to a σ-additive probability measure on a σ-algebra $\sigma\mathsf{U}$, that is, a minimal σ-algebra of subsets of X containing U. If $J: X_j \to X_j$ is a $\mathbf{K}$-linear operator diagonal in the basis $\{e_{k,j} : k\}$, then for z such that $z(x) = z_j(\phi_j(x))$ for each $x \in X$ and a symmetric non-negative definite operator F as in § 5

(i) $F(v_q(z\circ J), v_q(z\circ J)) = E(v_q(z), v_q(z))$, where

(ii) $E_{k,l} = F_{k,l}|J_{k,k}|^{q/2}|J_{l,l}|^{q/2}$ for each $k,l \in \alpha_j$. If $F \in L_a$ (that is, $F^a \in L_1$) and $J \in L_q$ (that is, $diag(v_1(J_{l,l}) : l) \in L_q$), then

(iii) $E \in L_{aq/(a+q)}$ for each $a > 0$ (see Theorem 8.2.7 [Pie65]). In particular, taking a tending to ∞ and $F = I$ we get $E \in L_q$, since L_∞ is the space of bounded linear operators. Using the orthonormal bases in X_j for each j we get the embedding of X_j into its topologically dual space X_j^* of all continuous $\mathbf{K}$-linear functionals on X_j. For each $z \in X^*$ there exists a non-Archimedean direct sum decomposition $X = X_z \oplus ker(z)$, where X_z is a one dimensional over $\mathbf{K}$ subspace in X. Therefore, the set $\mathsf{D}_{q,B,X}$ of functionals z on X from § 5 separates points of X. More generally consider in each X_j a sequence of projection operators $P_{V_{n,j}}$ on subspaces $V_{n,j} := \{ {}_{i,j}z : i = 1,\dots,n\}$, where $\{ {}_{i,j}z : i \in \mathbf{N}\}$ is the orthonormal basis in X_j. Then consider J in this new basis and the transition operator from the standard basis to the new one. The composition of these two operators generates the corresponding operator C on Y which is in general non-diagonal. If for a given one dimensional over $\mathbf{K}$ subspace W in X it is the equality $B(v_q(z), v_q(z)) = 0$ for each $z \in W$, then the projection μ_W of μ is the atomic measure with one atom being a singleton. If $B \in L_1$, then $B(v_q(z), v_q(z))$ and hence $\hat{\mu}(z)$ is correctly defined for each $z \in \mathsf{D}_{q,B,X}$.

It remains to establish that μ is σ-additive if and only if $J \in L_q(c_0(\omega_0,\mathbf{K}))$ and $\gamma \in c_0(\omega_0,\mathbf{K})$.

We have

$$\mu_j(\mathbf{K}\setminus B(\mathbf{K},0,r)) \le C \int_{x\in\mathbf{K},|x|>r} \exp(-|x/\zeta_j|^q)|\zeta_j|^{-1}v(dx)$$

$$\le C_1 \int_{y\in\mathbf{R},|y|>r} \exp(-|y|^q|\zeta_j|^{-q})|\zeta_j|^{-1}dy,$$

where $C > 0$ and $C_1 > 0$ are constants independent from ζ_j for $b_0 > p^3$ and each $r > b_0$, $1 \le q < \infty$ is fixed (see also the proof of Lemma 2.8 above and Theorem II.2.1 [DF91]). Evidently, $g(\gamma)$ is correctly defined for each $g \in c_0(\omega_0,\mathbf{K})^*$ if and only if $\gamma \in c_0(\omega_0,\mathbf{K})$. In this case the character $\chi_{g(\gamma)} : \mathbf{K} \to \mathbf{C}$ is defined and $\chi_{g(\gamma)} = \prod_{j=1}^\infty \chi_{g_j\gamma_j}$. Due to Lemma 2.3 above, if $J \in L_q(c_0)$ and $\gamma \in c_0(\omega_0,\mathbf{K})$, then μ is σ-additive.

Let $0 \ne g \in c_0^*$. Since $\mathbf{K}$ is the local field there exists $x_0 \in c_0$ such that $|g(x_0)| = \|g\|$ and $\|x_0\| = 1$. Put $g_j := g(e_j)$. Then $\|g\| \le \sup_j |g_j|$, since $g(x) = \sum_j x^j g_j$, where $x = x^j e_j := \sum_j x^j e_j$ with $x^j \in \mathbf{K}$. Consequently, $\|g\| = \sup_j |g_j|$. We enumerate the standard orthonormal basis $\{e_j : j \in \mathbf{N}\}$ such that $|g_1| = \|g\|$. There exists an operator E on c_0 with matrix elements $E_{i,j} = \delta_{i,j}$ for each $i, j > 1$, $E_{1,j} = g_j$ for each $j \in \mathbf{N}$. Then $|det\, P_n E P_n| = \|g\|$ for each $n \in \mathbf{N}$, where P_n are the standard projectors on $span_{\mathbf{K}}\{e_1,\dots,e_n\}$ [LD02] (see also Appendix and comments to it). When $g \in \{e_j^* : j \in \omega_0\}$, then evidently, μ^g has the form given by Equation (iii), since $\mu_i(\mathbf{K}) = 1$ for each $i \in \omega_0$, where $e_j^*(e_i) = \delta_{i,j}$ for each i, j.

Suppose now that $J \notin L_q(c_0)$. For this we consider $\mu^g(\mathbf{K}\setminus B(\mathbf{K},0,r)) \ge \sum_j \int_{x\in\mathbf{K},|x|>r} C \exp(-|x/\zeta_j|^q)|\zeta_j|^{-1}v(dx)$, where $g = (1,1,1,\dots) \in c_0^* = l^\infty(\omega_0,\mathbf{K})$. On the other hand, there exists a constant $C_2 > 0$ such that for $b_0 > p^3$ and each $r > b_0$ there is the

following inequality: $\int_{x\in\mathbf{K},|x|>r} C\ \exp(-|x/\zeta_j|^q)|\zeta_j|^{-1}\nu(dx) \geq$
$C_2[\int_r^\infty \exp(-|y|^q|\zeta_j|^{-q})|\zeta_j|^{-1}dy + \int_{-\infty}^{-r}\exp(-|y|^q|\zeta_j|^{-q})|\zeta_j|^{-1}dy]$. From the estimates of Lemma II.1.1 [DF91] and using the substitution $z = y^{1/2q}$ for $y > 0$ and $z = (-y)^{1/2q}$ for $y < 0$ we get that μ^g is not σ-additive, consequently, μ is not σ-additive, since $P_g^{-1}(A)$ are cylindrical Borel subsets for each $A \in Bf(\mathbf{K})$, where $P_g z = g(z)$ is the induced projection on $\mathbf{K}$ for each $z \in c_0$.

7. Corollary. *A q-Gaussian measure μ from Proposition 6 with $Tr(B) < \infty$ is quasi-invariant and pseudo-differentiable for some $b \in \mathbf{C}$ relative to a dense subspace $J_\mu \subset M_\mu = \{a \in X : v_q(a) \in E^{1/2}(Y)\}$. Moreover, if B is diagonal, then each one-dimensional projection μ^g has the following characteristic functional:*

$$(i)\quad \hat{\mu}^g(h) = \exp\Big(-\Big(\sum_j \beta_j|g_j|^q\Big)|h|^q\Big)\chi_{g(\gamma)}(h),$$

where $g = (g_j : j \in \omega_0) \in c_0(\omega_0, \mathbf{K})^*$.

Proof. Using the projective limit reduce consideration to the Banach space X. The first statement follows from Theorems 3.12, 4.2 and 4.4 (see also [Lud00a]). To find M_μ consider $a \in X$ and the q-Gaussian measures $\mu(dz)$ and $\mu_a(dz) := \mu(-a+dz)$. Each Hellinger integral $H(\mu_{a,n}, \mu_n)$ has a value in $[0,1]$, hence $\prod_{n=1}^\infty H(\mu_{a,n}, \mu_n)$ either diverges to zero and $\mu_a \perp \mu$ or converges to a number $0 < \beta \leq 1$ and $\mu_a \sim \mu$ (see Theorem 3.3.1). Suppose that μ_a is not orthogonal to μ, then $\mu_a \sim \mu$ and there exists $\mu_a(dx)/\mu(dx) = \lim_{n\to\infty}\mu_a^{V_n}(dx^n)/\mu^{V_n}(dx^n) \in L^1(X, \mathcal{B}, \mu, \mathbf{C})$, where $V_n := span_{\mathbf{K}}(e_1, \dots, e_n)$, $n \in \mathbf{N}$, μ^{V_n} is the projection of μ on V_n, $x^n := ({}_1x, \dots, {}_nx)$, ${}_jx \in \mathbf{K}$ for each j, $x^n \in V^n$. But

$$\mu_a(dx)/\mu(dx) = \lim_{n\to\infty}[\mu_a(dx^n)/\lambda_n(dx^n)][\mu^{V_n}(dx^n)/\lambda_n(dx^n)]^{-1},$$

where λ_n is the Haar nonnegative measure, hence $\mu_a^{V_n}(dx^n)/\lambda(dx^n) \in L^1(V_n, \mathcal{B}_n, \lambda_n, \mathbf{C})$ for each n. Choose an orthonormal basis $({}_jz : j \in \mathbf{N})$ in X and an operator $G : X \to X$ such that $G\,{}_jz = {}_ja\,{}_jz$, ${}_ja \neq 0$ for each j, hence $\mu(G^{-1}dy)$ has the correlation operator CEC, where $y \in G(X)$, $G^{-1} : G(X) \to X$, C is defined by G and the transition operator from the standard orthonormal basis $(e_j : j)$ to $({}_jz : j)$ (see also § II.6.21).

It is possible to take G such that CEC is the bounded continuous operator on Y and there exists the bounded continuous operator $(CEC)^{-1}$ on Y. The Fourier operator is unitary on $L^2(X, \mathcal{B}, \mu, \mathbf{C})$. Therefore, the existence of $\hat{\mu}_a$ relative to the measure μ implies $v_q(a) \in E^{1/2}(Y)$. Since $v_q(\xi a) = v_q(\xi)v_q(a)$ and $v_q(a_j + b_j) \leq \max(v_q(a_j), v_q(b_j))$ for each $\xi \in \mathbf{K}$ and $a, b \in X$, $a = \sum_j a_j e_j$, then the family of all such $a \in X$ with $v_q(a) \in E^{1/2}(Y)$ is the $\mathbf{K}$-linear subspace.

Vice versa, if $v_q(a) \in E^{1/2}(Y)$, then the proof above shows that there exists $\hat{\mu}_a$ relative to μ and hence there exists the limit
$\lim_{n\to\infty}[\mu_a^{V_n}(dx^n)/\lambda_n(dx^n)][\mu^{V_n}(dx^n)/\lambda_n(dx^n)]^{-1} = \mu_a(dx)/\mu(dx)$.

For the verification of Formula (i) it is sufficient at first to consider the measure μ on the algebra U^{P} of cylindrical subsets in c_0. Then for each projection μ^g, where $g \in span_{\mathbf{K}}(e_1, \dots, e_m)^*$, we have:

$$\hat{\mu}^g(h) = \int_{\mathbf{K}} \cdots \int_{\mathbf{K}} \chi_e(hz)\mu_1(dx_1)\cdots\mu_m(dx_m),$$

where $e = (1, \dots, 1) \in \mathbf{Q_p^n}$, $h \in \mathbf{K}$, $n := dim_{\mathbf{Q_p}}\mathbf{K}$, $x^i \in \mathbf{K}e_i$, $z = g(x)$, $x = (x^1, \dots, x^m)$, consequently, $\hat{\mu}^g(h) = \prod_{i=1}^m \hat{\mu}_i(hg_i)$, since $\chi_e(hg(x)) = \prod_{i=1}^m \chi_e(h_i g_i x^i)$ for each $x \in span_{\mathbf{K}}(e_1, \dots, e_m)$. Let J be a subspace of all $x \in X$ such that $v_q(x) \in B^{1/2}(\mathsf{D}_{B,Y})$. Since $\mathsf{D}_{B,Y}$ is $\mathbf{R}$-linear and $v_q(az) = |a|^{q/2} v_q(z)$ for each $a \in \mathbf{K}$, $v_q(y+z) \le \max(v_q(y), v_q(z))$, then J is $\mathbf{K}$-linear and μ is quasi-invariant relative to J. In view of the Parceval-Steklov equality and definition of pseudo-differentiabilty and convergence of $\int_{\mathbf{K}} \exp(-a(v_q(x))^2)|x|^m v(dx) < \infty$ for each $m \ge 0$ and each $a > 0$, where v is a nonnegative nontrivial Haar measure on $\mathbf{K}$ it follows, that μ is pseudo-differentiable for each $b \in \mathbf{C}$ with $Re(b) > 0$.

Since $B \in L_q$, then μ is the Radon measure, consequently, the continuation of μ from U^{P} produces μ on the Borel σ-algebra of c_0, hence $\lim_{m\to\infty} \hat{\mu}^{Q_m g}(h) = \hat{\mu}^g(h)$, where Q_m is the natural projection on $span_{\mathbf{K}}(e_1, \dots, e_m)^*$ for each $m \in \mathbf{N}$ such that $Q_m(g) = (g_1, \dots, g_m)$. Using expressions of $\hat{\mu}_i$ we get Formula (i). From this it follows, that if $B \in L_q$, then $\hat{\mu}(g)$ exists for each $g \in c_0^*$ if and only if $\gamma \in c_0$, since $\hat{\mu}^g(h) = \hat{\mu}(gh)$ for each $h \in \mathbf{K}$ and $g \in c_0^*$.

8. Corollary. *Let X be a complete locally* **K***-convex space of separable type over a local field* **K***, then for each constant $q > 0$ there exists a nondegenerate symmetric positive definite operator $B \in L_1$ such that a q-Gaussian measure is σ-additive on $Bf(X)$ and each its one dimensional over* **K** *projection is absolutely continuous relative to the nonnegative Haar measure on* **K**.

Proof. A space Y from § 4 corresponding to X is a separable locally $\mathbf{R}$-convex space. Therefore, Y in a weak topology is isomorphic with $\mathbf{R}^{\aleph_0}$ from which the existence of B follows. For each $\mathbf{K}$-linear finite dimensional over $\mathbf{K}$ subspace S a projection μ^S of μ on $S \subset X$ exists and its density $\mu^S(dx)/w(dx)$ relative to the non-negative nondegenerate Haar measure w on S is the inverse Fourier transform $F^{-1}(\hat{\mu}|_{S^*})$ of the restriction of $\hat{\mu}$ on S^* (see about the Fourier transform on non-Archimedean spaces § VII [VVZ94]). For such B each one dimensional projection of μ corresponding to $\hat{\mu}$ has a density that is a continuous function belonging to $L^1(\mathbf{K}, w, Bf(\mathbf{K}), \mathbf{R})$, where w denotes the nonnegative Haar measure on $\mathbf{K}$.

9. Proposition. *Let $\mu_{q,B,\gamma}$ and $\mu_{q,E,\delta}$ be two q-Gaussian measures with correlation operators B and E of class L_1, then there exists a convolution of these measures $\mu_{q,B,\gamma} * \mu_{q,E,\delta}$, which is a q-Gaussian measure $\mu_{q,B+E,\gamma+\delta}$.*

Proof. Since B and E are nonnegative, then $(B+E)(y,y) = B(y,y) + E(y,y) \ge 0$ for each $y \in Y$, that is, $B+E$ is nonnegative. Evidently, $B+E$ is symmetric. In view of [Pie65] $B+E$ is of class L_1. Therefore, $\mu_{q,B+E,\gamma+\delta}$ is the σ-additive q-Gaussian measure together with $\mu_{q,B,\gamma}$ and $\mu_{q,E,\delta}$ in accordance with Proposition 6. Moreover, $\mu_{q,B+E,\gamma+\delta}$ is defined on the σ-algebra $\sigma\mathsf{U}_{B+E}$ containing the union of σ-algebras $\sigma\mathsf{U}_B$ and $\sigma\mathsf{U}_E$ on which $\mu_{q,B,\gamma}$ and $\mu_{q,E,\delta}$ are defined correspondingly, since $ker(B+E) \subset ker(B) \cap ker(E)$. Since $\hat{\mu}_{q,B+E,\gamma+\delta} = \hat{\mu}_{q,B,\gamma}\hat{\mu}_{q,E,\delta}$, then $\mu_{q,B+E,\gamma+\delta} = \mu_{q,B,\gamma} * \mu_{q,E,\delta}$ (see Proposition A.12 in Appendix and use projective limits).

9.1. Remark and Definition. A measurable space (Ω, F) with a probability real-valued measure λ on a covering σ-algebra F of a set Ω is called a probability space and it is denoted by $(\Omega, \mathsf{F}, \lambda)$.

The random variable ξ induces a normalized measure $\nu_\xi(A) := \lambda(\xi^{-1}(A))$ in X and a new probability space (X, B, ν_ξ).

Let T be a set with a covering σ-algebra $\mathcal{R}$ and a measure $\eta : \mathcal{R} \to \mathbf{R}$. Denote by $L^q(T, \mathcal{R}, \eta, H)$ the completion of the set of all $\mathcal{R}$-step functions $f : T \to H$ relative to the following norm:

(1) $\|f\|_{\eta,q} := (\int_{t\in T}\|f(t)\|_H^q\eta(dt))^{1/q}$ for $1\le q<\infty$ and

(2) $\|f\|_{\eta,\infty} := ess-\sup_{\eta,t\in T}\|f(t)\|_H$, where H is a Banach space over **K**. For $0<q<1$ this is the metric space with the metric

(3) $\rho_q(f,g) := (\int_{t\in T}\|f(t)-g(t)\|_H^q\eta(dt))^{1/q}$.

If H is a complete locally **K**-convex space, then H is a projective limit of Banach spaces $H=\lim\{H_\alpha,\pi^\alpha_\beta,\Upsilon\}$, where Υ is a directed set, $\pi^\alpha_\beta : H_\alpha\to H_\beta$ is a **K**-linear continuous mapping for each $\alpha\ge\beta$, $\pi_\alpha : H\to H_\alpha$ is a **K**-linear continuous mapping such that $\pi^\alpha_\beta\circ\pi_\alpha = \pi_\beta$ for each $\alpha\ge\beta$ (see § 6.205 [NB85]). Each norm p_α on H_α induces a pre-norm $\tilde{p}_\alpha$ on H. If $f:T\to H$, then $\pi_\alpha\circ f =: f_\alpha : T\to H_\alpha$. In this case $L^q(T,\mathcal{R},\eta,H)$ is defined as a completion of a family of all step functions $f:T\to H$ relative to the family of pre-norms

(1′) $\|f\|_{\eta,q,\alpha} := (\int_{t\in T}\tilde{p}_\alpha(f(t))^q\eta(dt))^{1/q}$, $\alpha\in\Upsilon$, for $1\le q<\infty$ and

(2′) $\|f\|_{\eta,\infty,\alpha} := ess-\sup_{\eta,t\in T}\tilde{p}_\alpha(f(t))$, $\alpha\in\Upsilon$, or pseudo-metrics

(3′) $\rho_{q,\alpha}(f,g) := (\int_{t\in T}[\tilde{p}_\alpha(f(t)-g(t))]^q\eta(dt))^{1/q}$, $\alpha\in\Upsilon$, for $0<q<1$. Therefore, $L^q(T,\mathcal{R},\eta,H)$ is isomorphic with the projective limit $\lim\{L^q(T,\mathcal{R},\eta,H_\alpha),\pi^\alpha_\beta,\Upsilon\}$. For example, T may be a subset of **R**. If $T\subset$ **F** with a non-Archimedean field **F**, then we can consider the non-Archimedean time parameter also.

If T is a zero-dimensional T_1-space, then denote by $C^0_b(T,H)$ the Banach space of all continuous bounded functions $f:T\to H$ supplied with the norm:

(4) $\|f\|_{C^0} := \sup_{t\in T}\|f(t)\|_H<\infty$.

For a set T and a complete locally **K**-convex space H over **K** consider the product **K**-convex space $H^T := \prod_{t\in T}H_t$ in the product topology, where $H_t := H$ for each $t\in T$.

Then take on either $X := X(T,H) = L^q(T,\mathcal{R},\eta,H)$ or $X := X(T,H) = C^0_b(T,H)$ or on $X = X(T,H) = H^T$ a covering σ-algebra B such that $\mathsf{B}\supset Bf(X)$. Consider a random variable $\xi : \omega\mapsto\xi(t,\omega)$ with values in (X,B), where $t\in T$.

Consider T such that $card(T)>n$. For $X=C^0_b(T,H)$ or $X=H^T$ define $X(T,H;(t_1,\dots,t_n);(z_1,\dots,z_n))$ as a closed sub-manifold in X of all $f:T\to H$, $f\in X$ such that $f(t_1)=z_1,\dots,f(t_n)=z_n$, where $t_1,\dots,t_n$ are pairwise distinct points in T and $z_1,\dots,z_n$ are points in H. For $X=L^q(T,\mathcal{R},\eta,H)$ and pairwise distinct points $t_1,\dots,t_n$ in T with $supp(\eta)\supset\{t_1,\dots,t_n\}$ define $X(T,H;(t_1,\dots,t_n);$ $(z_1,\dots,z_n))$ as a closed sub-manifold which is the completion relative to the norm $\|f\|_{\eta,q}$ of a family of $\mathcal{R}$-step functions $f:T\to H$ such that $f(t_1)=z_1,\dots,f(t_n)=z_n$. In these cases $X(T,H;(t_1,\dots,t_n);(0,\dots,0))$ is the proper **K**-linear subspace of $X(T,H)$ such that $X(T,H)$ is isomorphic with $X(T,H;(t_1,\dots,t_n);(0,\dots,0))\oplus H^n$, since if $f\in X$, then $f(t)-f(t_1) =: g(t)\in X(T,H;t_1;0)$ (in the third case we use that $T\in\mathcal{R}$ and hence there exists the embedding $H\hookrightarrow X$). For $n=1$ and $t_0\in T$ and $z_1=0$ we denote $X_0 := X_0(T,H) := X(T,H;t_0;0)$.

9.2. Definitions. We define a (non-Archimedean) stochastic process $w(t,\omega)$ with values in H as a random variable such that:

(i) the differences $w(t_4,\omega)-w(t_3,\omega)$ and $w(t_2,\omega)-w(t_1,\omega)$ are independent for each chosen (t_1,t_2) and (t_3,t_4) with $t_1\ne t_2$, $t_3\ne t_4$, such that either t_1 or t_2 is not in the two-element set $\{t_3,t_4\}$, where $\omega\in\Omega$;

(ii) the random variable $\omega(t,\omega)-\omega(u,\omega)$ has a distribution $\mu^{F_{t,u}}$, where μ is a probability real-valued measure on $(X(T,H),\mathsf{B})$ from § 9.1, $\mu^g(A) := \mu(g^{-1}(A))$ for $g:X\to H$ such that $g^{-1}(\mathcal{R}_H)\subset\mathsf{B}$ and each $A\in\mathcal{R}_H$, $F_{t,u}(w) := w(t,\omega)-w(u,\omega)$ for each $w\in L^q(\Omega,\mathsf{F},\lambda;X)$, where $1 \le q \le \infty$, $\mathcal{R}_H$ is a covering σ-algebra of H such that $F^{-1}_{t,u}(\mathcal{R}_H) \subset \mathsf{B}$ for each $t \ne u$ in T;

(iii) we also put $w(0,\omega) = 0$, that is, we consider a $\mathbf{K}$-linear subspace $L^q(\Omega, \mathsf{F}, \lambda; X_0)$ of $L^q(\Omega, \mathsf{F}, \lambda; X)$, where $\Omega \ne \emptyset$, X_0 is the closed subspace of X as in § 9.1.

10. Definitions. Let B and q be as in § 6 and denote by $\mu_{q,B,\gamma,a}$ the corresponding q-Gaussian measure on H. Let ξ be a stochastic process with a real time $t \in T \subset \mathbf{R}$ (see Definition 9.2), then it is called a non-Archimedean q-Wiener process with real time, if

$(ii)'$ the random variable $\xi(t,\omega) - \xi(u,\omega)$ has a distribution $\mu_{q,(t-u)B,\gamma}$ for each $t \ne u \in T$.

Let ξ be a stochastic process with a non-Archimedean time $t \in T \subset \mathbf{F}$, where $\mathbf{F}$ is a local field, then ξ is called a non-Archimedean q-Wiener process with $\mathbf{F}$-time, if

(ii)" the random variable $\xi(t,\omega) - \xi(u,\omega)$ has a distribution $\mu_{q,ln[\chi_{\mathbf{F}}(t-u)]B,\gamma}$ for each $t \ne u \in T$, where $\chi_{\mathbf{F}} : \mathbf{F} \to S^1$ is a continuous character of $\mathbf{F}$ as the additive group.

11. Proposition. *For each given q-Gaussian measure a non-Archimedean q-Wiener process with real* ($\mathbf{F}$ *respectively*) *time exists.*

Proof. In view of Proposition 9 for each $t > u > b$ a random variable $\xi(t,\omega) - \xi(b,\omega)$ has a distribution $\mu_{q,(t-b)B,\gamma}$ for real time parameter. If t, u, b are pairwise different points in $\mathbf{F}$, then $\xi(t,\omega) - \xi(b,\omega)$ has a distribution $\mu_{q,ln[\chi_{\mathbf{F}}(t-b)]B,\gamma}$, since $ln[\chi_{\mathbf{F}}(t-u)] + ln[\chi_{\mathbf{F}}(u-b)] = ln[\chi_{\mathbf{F}}(t-b)]$. This induces the Markov quasi-measure $\mu^{(q)}_{x_0,\tau}$ on $(\prod_{t\in T}(H_t, \mathsf{U}_t))$, where $H_t = H$ and $\mathsf{U}_t = Bf(H)$ for each $t \in T$ (see § VI.1.1 [DF91] and § 3 in [Lud0321]). Therefore, the Chapman-Kolmogorov equation is accomplished:

$$P(b,x,t,A) = \int_H P(b,x,u,dy)P(u,y,t,A)$$

for each $A \in Bf(H)$. An abstract probability space $(\Omega, \mathsf{F}, \lambda)$ exists due to the Kolmogorov theorem, hence the corresponding space L^r exists. Therefore, conditions of Definitions 10 are satisfied (see also 4.1 [Lud0321]).

12. Proposition. *Let ξ be a q-Gaussian process with values in a Banach space $H = c_0(\alpha, \mathbf{K})$ a time parameter $t \in T$ and a positive definite correlation operator B of trace class and $\gamma = 0$, where $card(\alpha) \le \aleph_0$, either $T \subset \mathbf{R}$ or $T \subset \mathbf{F}$. Then either*

$$(i) \quad \lim_{N\in\alpha} M_t \|(v_q(e^1(\xi(t,\omega)), \dots, v_q(e^N(\xi(t,\omega)))\|^2_{l_2} = tTr(B) \text{ or}$$

$$(ii) \quad \lim_{N\in\alpha} M_t \|(v_q(e^1(\xi(t,\omega)), \dots, v_q(e^N(\xi(t,\omega))\|^2_{l_2} = [ln(\chi_{\mathbf{F}}(t))]Tr(B) \text{ respectively.}$$

Proof. At first we consider moments of a q-Gaussian measure $\mu_{q,B,\gamma}$. We define moments $m^q_k(e^{j_1}, \dots, e^{j_k}) := \int_H v_{2q}(e^{j_1}(x)) \cdots v_{2q}(e^{j_k}(x))\mu_{q,B,\gamma}(dx)$ for linear continuous functionals $e^{j_1}, \dots, e^{j_k}$ on H such that $e^l(e_j) = \delta^l_j$, where in our previous notation $\{e_j : j \in \alpha\}$ is the standard orthonormal base in H.

Consider partial pseudo-differential operators ${}_P\partial^u_j$ given by the equation

$$(iii) \quad {}_P\partial^u_j \psi(x) := \mathsf{F}^{-1}_j(|\tilde{x}_j|^u_{\mathbf{K}} \hat{\psi}(\tilde{x}))(x),$$

where the norm $|b|_{\mathbf{K}} = mod_{\mathbf{K}}(b)$ on $\mathbf{K}$ is chosen coinciding with the modular function associated with the nonnegative nondegenerate Haar measure w on $\mathbf{K}$ (about the modular function see [Wei73]), $u \in \mathbf{C} \setminus \{-1\}$, $\hat{\psi} := \mathsf{F}_j(\psi)$ is the Fourier transform of ψ by a variable $x_j \in \mathbf{K}$ such that F_j is defined relative to the Haar measure w on $\mathbf{K}$ [VVZ94]. From the change of variables formula $\int_{\mathbf{K}} f(ax+b)g(x)w(dx) = \int_{\mathbf{K}} f(y)g((y-b)/a)|a|_{\mathbf{K}}^{-1}w(dy)$ for each f and $g \in L^2(\mathbf{K}, Bf(\mathbf{K}), w, \mathbf{C})$, $a \neq 0$ and $b \in \mathbf{K}$, also the Fubini theorem and the Fourier transform on $\mathbf{K}$ it follows that $f_{-\alpha} * f_{u+1} = f_{u+1-\alpha}$ for $u \neq \alpha$ and $\Gamma_{\mathbf{K}}(u+1)|\xi_j|_{\mathbf{K}}^{-u-1} = \mathsf{F}(|x_j|^u)$, where $f_u(x_j) := |x_j|^{u-1}/\Gamma_{\mathbf{K}}(u)$, $\Gamma_{\mathbf{K}}$ is the non-Archimedean gamma function, $\Gamma_{\mathbf{K}}(u) := \int_{\mathbf{K}} |z|_{\mathbf{K}}^{u-1}\chi(z)w(dz)$, $\chi : \mathbf{K} \to S^1$ is the character of $\mathbf{K}$ as the additive group such that $\chi(z) := \prod_{j=1}^m \chi_p(z'_j)$, $z'_j \in \mathbf{Q_p}$, $z = (z'_1, \dots, z'_m) \in \mathbf{K}$ for $\mathbf{K}$ considered as the $\mathbf{Q_p}$-linear space, $m \in \mathbf{N}$, $dim_{\mathbf{Q_p}}\mathbf{K} = m$, $\chi_p : \mathbf{Q_p} \to S^1$ is the standard character such that $\chi_p(y) := \exp(2\pi i \{y\}_p)$, $\{y\}_p := \sum_{l<0} a_l p^l$ for $|y|_{\mathbf{Q_p}} > 1$ and $\{y\}_p = 0$ for $|y|_{\mathbf{Q_p}} \leq 1$, $y = \sum_l a_l p^l$, $a_l \in \{0, 1, \dots, p-1\}$, $l \in \mathbf{Z}$, $\min(l : a_l \neq 0) =: ord_p(y) > -\infty$. Therefore, ${}_P\partial_j^u |x_j|^n = |x_j|^{n-u}\Gamma_{\mathbf{K}}(n)/\Gamma_{\mathbf{K}}(n-u)$, where $n \in \mathbf{C} \setminus \{-1\}$. A function ψ for which ${}_P\partial_j^u \psi$ exists is called pseudo-differentiable of order u by variable x_j.

From $m_k^{q/2}(e^{j_1}, \dots, e^{j_k}) = \mathsf{F}^{-1}(|x_{j_1}|^{q/2} \cdots |x_{j_k}|^{q/2}\mathsf{F}(\mu))(0)$, since $\mathsf{F}(hg) = \mathsf{F}(h) * \mathsf{F}(g)$ for functions h and g in the Hilbert space $L^2(\mathbf{K}, Bf(\mathbf{K}), w, \mathbf{C})$ it follows that $m_{2k}^{q/2}(e^{j_1}, \dots, e^{j_{2k}}) = {}_P\partial_{j_1}^{q/2} \cdots {}_P\partial_{j_{2k}}^{q/2}\hat{\mu}(0) = ([{}_PD^{q/2}]^{2k}\hat{\mu}(0)).(e_{j_1}, \dots, e_{j_{2k}})$, where ${}_PD^{q/2}$ is a $\mathbf{K}$-linear pseudo-differential operator by $x \in H$ such that

$$({}_PD^{q/2}\psi(x)).e_j := {}_P\partial_j^{q/2}\psi(x).$$

Then

(iv) $m_{2n}^{q/2}(e^{j_1}, \dots, e^{j_{2n}}) = (-1)^n (n!)^{-1} [{}_PD^{q/2}]^{2n}[B(v_q(z), v_q(z)]^n.(e_{j_1}, \dots, e_{j_{2n}})$
$= (n!)^{-1} \sum_{\sigma \in \Sigma_{2n}} B_{\sigma(j_1),\sigma(j_2)} \cdots B_{\sigma(j_{2n-1}),\sigma(j_{2n})}$,

since $\gamma = 0$ and $\chi_\gamma(z) = 1$, where Σ_k is the symmetric group of all bijective mappings σ of the set $\{1, \dots, k\}$ onto itself, $B_{l,j} := B(e_j, e_l)$, since $Y^* = Y$ for $Y = l_2(\alpha, \mathbf{R})$. Therefore, for each $B \in L_1$ and $A \in L_\infty$ we have $\int_H A(v_q(x), v_q(x))\mu_{q,B,0}(dx) = \lim_{N \in \alpha} \sum_{j=1}^N \sum_{k=1}^N A_{j,k} m_2^{q/2}(e_j, e_k) = Tr(AB)$.

In particular for $A = I$ and $\mu_{q,tB,0}$ corresponding to the transition measure of $\xi(t, \omega)$ we get Formula (i) for a real time parameter, using $\mu_{q,ln[\chi_{\mathbf{F}}(t)]B,0}$ we get Formula (ii) for a time parameter belonging to $\mathbf{F}$, since $\xi(t_0, \omega) = 0$ for each ω.

13. Corollary. *Let $H = \mathbf{K}$ and ξ, $B = 1$, γ be as in Proposition* 12, *then*

$$(i) \quad M\left(\int_{t \in [a,b]} \phi(t,\omega)|d\xi(t,\omega)|_{\mathbf{K}}^q\right) = M\left[\int_a^b \phi(t,\omega)dt\right]$$

for each $a < b \in T$ with real time, where $\phi(t,\omega) \in L^2(\Omega, \mathsf{U}, \lambda, C_0^0(T, \mathbf{R}))$ $\xi \in L^q(\Omega, \mathsf{U}, \lambda, X_0(T, \mathbf{K}))$, $(\Omega, \mathsf{U}, \lambda)$ is a probability measure space.

Proof. Since $\int_{t \in [a,b]} \phi(t,\omega)|d\xi(t,\omega)|_{\mathbf{K}}^q$
$= \lim_{\max_j (t_{j+1}-t_j) \to 0} \sum_{j=1}^N \phi(t_j, \omega)|\xi(t_{j+1}, \omega) - \xi(t_j, \omega)|_{\mathbf{K}}^q$ for λ-almost all $\omega \in \Omega$, then applying Formula 12.(i) to each $|\xi(t_{j+1}, \omega) - \xi(t_j, \omega)|_{\mathbf{K}}^q$ and taking the limit by finite partitions $a = t_1 < t_2 < \dots < t_{N+1} = b$ of the segment $[a, b]$ we get Formula 13.(i).

14. Remarks. In the classical case with $q = 2$ and $\mathbf{R}$ instead of $\mathbf{K}$ there is analogous formula $M([\int_{t \in [a,b]} \phi(t,\omega)dB_t(\omega)]^2) = M[\int_a^b \phi(t,\omega)^2 dt]$ known as the It$\hat{o}$ formula (see the classical case in [Bou63-69, DF91]). Another analogs of the It$\hat{o}$ formula were given in [Lud0341] (see also comments in § 1.7). Certainly it is impossible to get in the non-Archimedean case all the same properties of Gaussian measures and Wiener process (Brownian motion) as in the classical case. Therefore, there are different possibilities for seeking non-Archimedean analogs of Gaussian measures and Wiener processes depending on a set of properties supplied with these objects. Giving our definitions we had the intention to take into account the most important properties.

Since $\mathsf{F}(\chi_\gamma)(y) = \delta(y-\gamma)$ and $[\delta(y-\gamma)*h(y)](x) = h(x-\gamma)$ for any continuous function h, then $\int_H |x_{j_1}-\gamma_{j_1}|^{q/2}\cdots|x_{j_k}-\gamma_{j_k}|^{q/2} d\mu_{q,B,\gamma} = \int_H |x_{j_1}|^{q/2}\cdots|x_{j_k}|^{q/2} d\mu_{q,B,0}$, consequently, γ plays in some sense the mean value role.

If $A > 0$ on $Y = l_2(\alpha,\mathbf{K})$, then

$\mu_{q,B,0}\{x : A(v_q(x),v_q(x)) \ge 1\} \le Tr(AB)$ and

$\mu_{q,B,0}\{x : |A(v_q(x),v_q(x)) - Tr(AB)| \le c(Tr(AB))^{1/2}\} \ge 1-2\|AB\|/c^2$ for each $c > 0$ due to the Chebyshev inequality and Formula 12.(iv).

15. Definitions and Notes. Consider a pseudo-differential operator on $H = c_0(\alpha,\mathbf{K})$ such that

$$(i)\quad \mathsf{A} = \sum_{0\le k\in\mathbf{Z};\, j_1,\dots,j_k\in\alpha} (-i)^k b^k_{j_1,\dots,j_k}\ {}_P\partial_{j_1}\cdots {}_P\partial_{j_k},$$

where $b^k_{j_1,\dots,j_k} \in \mathbf{R}$, ${}_P\partial_{j_k} := {}_P\partial^1_{j_k}$. If there exists $n := \max\{k : b^k_{j_1,\dots,j_k} \ne 0, j_1,\dots,j_k \in \alpha\}$, then n is called an order of A, $Ord(\mathsf{A})$. If $\mathsf{A} = 0$, then by definition $Ord(\mathsf{A}) = 0$. If there is not any such finite n, then $Ord(\mathsf{A}) = \infty$. We suppose that the corresponding form $\tilde{A}$ on $\bigoplus_k Y^k$ is continuous into $\mathbf{C}$, where

$$(ii)\quad \tilde{A}(y) = -\sum_{0\le k\in\mathbf{Z};\, j_1,\dots,j_k\in\alpha} (-i)^k b^k_{j_1,\dots,j_k} y_{j_1}\cdots y_{j_k},$$

$y \in l_2(\alpha,\mathbf{R}) =: Y$. If $\tilde{A}(y) > 0$ for each $y \ne 0$ in Y, then A is called strictly elliptic pseudo-differential operator. The phase multiplier $(-i)^k$ is inserted into the definition of A for in the definition of ${}_P\partial_j$ it was omitted in comparison with the classical case.

Let X be a complete locally $\mathbf{K}$-convex space, let Z be a complete locally $\mathbf{C}$-convex space. For $0 \le n \in \mathbf{R}$ a space of all functions $f : X \to Z$ such that $f(x)$ and $({}_PD^k f(x)).(y^1,\dots,y^{l(k)})$ are continuous functions on X for each $y^1,\dots,y^{l(k)} \in \{e^1,e^2,e^3,\dots\} \subset X^*$, $l(k) := [k] + sign\{k\}$ for each $k \in \mathbf{N}$ such that $k \le [n]$ and also for $k = n$ is denoted by ${}_PC^n(X,Z)$ and $f \in {}_PC^n(X,Z)$ is called n times continuously pseudo-differentiable, where $[n] \le n$ is an integer part of n, $1 > \{n\} := n-[n] \ge 0$ is a fractional part of n, $sign(b) = 1$ for each $b > 0$, $sign(0) = 0$, $sign(b) = -1$ for $b < 0$. Then ${}_PC^\infty(X,Z) := \bigcap_{n=1}^\infty {}_PC^n(X,Z)$ denotes a space of all infinitely pseudo-differentiable functions.

16. Theorem. *Let* A *be a strictly elliptic pseudo-differential operator on* $H = c_0(\alpha,\mathbf{K})$, $card(\alpha) \le \aleph_0$, *and let* $t \in T = [0,b] \subset \mathbf{R}$. *Suppose also that* $u_0(x-y) \in L^2(H,Bf(H),\mu_{t\tilde{A}},\mathbf{C})$ *for each marked* $y \in H$ *as a function by* $x \in H$, $u_0(x) \in {}_PC^{Ord(\mathsf{A})}(H,\mathbf{C})$. *Then the non-Archimedean analog of the Cauchy problem*

$$(i)\quad \partial u(t,x)/\partial t = \mathsf{A}u,\quad u(0,x) = u_0(x)$$

has a solution given by

$$(ii)\quad u(t,x) = \int_H u_0(x-y)\mu_{t\tilde{A}}(dy),$$

where $\mu_{t\tilde{A}}$ *is a* σ*-additive Borel measure on* H *with a characteristic functional* $\hat{\mu}_{t\tilde{A}}(z) := \exp[-t\tilde{A}(v_2(z))]$.

Proof. In accordance with § § 4 and 15 we have $Y = l_2(\alpha,\mathbf{R})$. In view of the conditions of this theorem the function $\exp[-t\tilde{A}(v_2(z))]$ is continuous on $H \hookrightarrow H^*$ for each $t \in \mathbf{R}$ such that the family H of continuous $\mathbf{K}$-linear functionals on H separates points in H. In view of the Minlos-Sazonov theorem 2.35 above it defines a σ-additive Borel measure on H for each $t > 0$ and hence for each $t \in (0,b]$. The functional $\tilde{A}$ on each ball of radius $0 < R < \infty$ in Y is a uniform limit of its restrictions $\tilde{A}|_{\bigoplus_k [span_{\mathbf{K}}(e_1,\dots,e_n)]^k}$, when n tends to the infinity, since $\tilde{A}$ is continuous on $\bigoplus_k Y^k$. Since $u_0(x-y) \in L^2(H,Bf(H),\mu_{t\tilde{A}},\mathbf{C})$ and a space of cylindrical functions is dense in the latter Hilbert space, then due to the Parceval-Steklov equality and the Fubini theorem it follows that $\lim_{P\to I} \mathsf{F}_{Px}u_0(Px))\hat{\mu}_{t\tilde{A}}(y+Px)$ converges in $L^2(H,Bf(H),\mu_{t\tilde{A}},\mathbf{C})$ for each t, since $\mu_{t_1\tilde{A}} * \mu_{t_2\tilde{A}} = \mu_{(t_1+t_2)\tilde{A}}$ for each t_1, t_2 and $t_1+t_2 \in T$, where P is a projection on a finite dimensional over $\mathbf{K}$ subspace $H_P := P(H)$ in H, $H_P \hookrightarrow H$, P tends to the unit operator I in the strong operator topology, $\mathsf{F}_{Px}u_0(Px)$ denotes a Fourier transform by the variable $Px \in H_P$. Consider a function $v := \mathsf{F}_x(u)$, then $\partial v(t,x)/\partial t = -\tilde{A}(v_2(x))v(t,x)$, consequently, $v(t,x) = v_0(x)\exp[-t\tilde{A}(v_2(x))]$. From $u(t,x) = \mathsf{F}_x^{-1}(v(t,x))$, where as above $\mathsf{F}_x(u)$ denotes the Fourier transform by the variable $x \in H$ such that $\mathsf{F}_x(u(t,x)) = \lim_{n\to\infty} \mathsf{F}_{x_1,\dots,x_n}u(t,x)$. Therefore, $u(t,x) = u_0(x) * \mathsf{F}_x^{-1}(\hat{\mu}_{t\tilde{A}}) = \int_H u_0(x-y)\mu_{t\tilde{A}}(dy)$, since $u_0(x-y) \in L^2(H,Bf(H),\mu_{t\tilde{A}},\mathbf{C})$ and $\mu_{t\tilde{A}}$ is the bounded measure on $Bf(H)$ and $|\int_H u_0(x-y)\mu_{t\tilde{A}}(dy)| \le (\int_H |u_0(x-y)|^2\mu_{t\tilde{A}}(dy))\mu_{t\tilde{A}}(H) < \infty$.

17. Note. In the particular case of $Ord(\mathsf{A}) = 2$ and $\tilde{A}$ corresponding to the Laplace operator, that is, $\tilde{A}(y) = \sum_{l,j} g_{l,j}y_l y_j$, Equation 12.$(i)$ is (the non-Archimedean analog of) the heat equation on H. This provides the interpretation of the 2-Gaussian measure $\mu_{t\tilde{A}} = \mu_{2,t\tilde{A},0}$. For $dim_{\mathbf{K}}H < \infty$ the density $\mu_{t\tilde{A}}(dx)/w(dx)$ is called the heat kernel, where w is the nonnegative nondegenerate Haar measure on H.

For $Ord(\mathsf{A}) < \infty$ the form $\tilde{A}_0(y)$ corresponding to sum of terms with $k = Ord(\mathsf{A})$ in Formula 15.(ii) is called the principal symbol of operator A. If $\tilde{A}_0(y) > 0$ for each $y \ne 0$, then A is called an elliptic pseudo-differential operator. Evidently, Theorem 16 is true for elliptic A of $Ord(\mathsf{A}) < \infty$, since $\exp[-t\tilde{A}(v_2(z))]$ is the bounded continuous real-valued positive definite function.

18. Remark and Definitions. Let linear spaces X over $\mathbf{K}$ and Y over $\mathbf{R}$ be as in § 4 and B be a symmetric nonnegative definite (bilinear) operator on a dense $\mathbf{R}$-linear subspace $\mathsf{D}_{B,Y}$ in Y^*. A quasi-measure μ with a characteristic functional

$$\hat{\mu}(\zeta,x) := \exp[-\zeta B(v_q(z),v_q(z))]\chi_\gamma(z)$$

for a parameter $\zeta \in \mathbf{C}$ with $Re(\zeta) \ge 0$ defined on $\mathsf{D}_{q,B,X}$ is called a complex-valued Gaussian measure and is denoted by $\mu_{q,\zeta B,\gamma}$ also, where $\mathsf{D}_{q,B,X} := \{z \in X^* :$ there exists $j \in \Upsilon$ such that $z(x) = z_j(\phi_j(x))\ \forall x \in X,\ v_q(z) \in \mathsf{D}_{B,Y}\}$.

19. Proposition. *Let* $X = \mathsf{D}_{q,B,X}$ *and* B *be positive definite, then for each function* $f(z) := \int_X \chi_z(x)\nu(dx)$ *with a complex-valued measure* ν *of finite variation and each* $Re(\zeta) > 0$ *there exists*

$$(i) \quad \int_X f(z)\mu_{\zeta B}(dz) = \lim_{P\to I}\int_X f(Pz)\mu_{\zeta B}^{(P)}(dz)$$
$$= \int_X \exp(-\zeta B(v_q(z),v_q(z)))\chi_\gamma(z)\nu(dz),$$

where $\mu^{(P)}(P^{-1}(A)) := \mu(P^{-1}(A))$ *for each* $A \in Bf(X_P)$, $P : X \to X_P$ *is a projection on a* **K**-*linear subspace* X_P, *a convergence* $P \to I$ *is considered relative to a strong operator topology.*

Proof. A complex-valued measure ν can be presented as $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$, where ν_j are nonnegative measures, $j = 1, \dots, 4$, $i = (-1)^{1/2}$. Using the projective limit decomposition of X and § 2.27 above we get that

$$(ii) \quad \int_X f(z)\mu_{\zeta B}(dz) = \lim_{P\to I} \int_X f(Pz)\mu^{(P)}_{\zeta B}(dz).$$

On the other hand, for each finite dimensional over **K** subspace X_P

$$(iii) \quad \int_X f(Pz)\mu^{(P)}_{\zeta B}(dz) = \int_{X_P} \{\exp(-\zeta B(v_q(z), v_q(z)))\chi_\gamma(z)\}|_{X_P}\nu^{X_P}(dz).$$

Since each measure ν_j is non-negative and finite, then due to Lemma 2.3 and § 2.5 above there exists the limit

$$\lim_{P\to I} \int_{X_P} \{\exp[-\zeta B(v_q(z), v_q(z))]\chi_\gamma(z)\}|_{X_P}\nu^{X_P}(dz)$$

$$= \int_X \exp(-\zeta B(v_q(z), v_q(z)))\chi_\gamma(z)\nu(dz).$$

20. Proposition. *If conditions of Proposition* 19 *are satisfied and*

$$(i) \quad \int_{X_P} |f(Px)| w^{X_P}(dx) < \infty$$

for each finite dimensional over **K** *subspace* X_P *in* X, *then Formula* $19.(i)$ *is accomplished for* ζ *with* $Re(\zeta) = 0$, *where* w^{X_P} *is a non-negative nondegenerate Haar measure on* X_P.

Proof. The finite dimensional over **K** distribution

$$\mu^{X_P}_{q,iB,\gamma}/w^{X_P}(dx) = \mathsf{F}^{-1}(\hat{\mu}_{q,iB,\gamma}|_{X_P})$$

is locally w^{X_P}-integrable, but does not belong to the space $L^1(X_P, Bf(X_P), w^{X_P}, \mathbf{C})$. In view of Condition $20.(i)$ above and the Fubini theorem and using the Fourier transform of generalized functions (see § VII.3 [VVZ94]) we get Formulas $19.(ii, iii)$. Taking the limit by $P \to I$ we get Formula $19.(i)$ in the sense of distributions.

21. Remark. A measure $\mu_{q,iB,\gamma}$ is the non-Archimedean analog of the Feynman quasi-measure. Put

$$(i) \quad {}_F\int_X f(x)\mu_{q,iB,\gamma}(dx) := \lim_{\zeta\to i} \int_X f(x)\mu_{q,\zeta B,\gamma}(dx)$$

if such limit exists. If conditions of Proposition 19 are satisfied, then $\psi(\zeta) := \int_X f(x)\mu_{q,\zeta B,\gamma}(dx)$ is the holomorphic function on $\{\zeta \in \mathbf{C} : Re(\zeta) > 0\}$ and it is continuous on $\{\zeta \in \mathbf{C} : Re(\zeta) \geq 0\}$, consequently,

$$(ii) \quad {}_F\int_X f(x)\mu_{q,iB,\gamma}(dx) = \int_X \exp\{-iB(v_q(x), v_q(x))\}\chi_\gamma(x)\nu(dx).$$

Above were defined non-Archimedean analogs of Gaussian measures with specific properties, but usual Gaussian measures does not exist on non-Archimedean spaces as the following theorem shows.

22. Theorem. *Let X be a Banach space of separable type over a locally compact field* $\mathbf{K}$. *If on $Bf(X)$ there exists a nontrivial real-valued (probability) usual Gaussian measure, then* $\mathbf{K} = \mathbf{R}$.

Proof. Let μ be a nontrivial usual Gaussian real-valued measure on $Bf(X)$. Then by the definition its characteristic functional $\hat{\mu}$ must be positive definite complex-valued function such that $\hat{\mu}(0) = 1$, $\lim_{|y|\to\infty} \hat{\mu}(y) = 0$ for each $y \in X^* \setminus \{0\}$, where X^* is the topological conjugate space to X of all continuous $\mathbf{K}$-linear functionals $f : X \to \mathbf{K}$. Moreover, there exist a $\mathbf{K}$-bilinear functional g and a compact non-degenerate $\mathbf{K}$-linear operator $T : X^* \to X^*$ with $ker(T) = \{0\}$ and a marked vector $x_0 \in X$ such that $\hat{\mu}_{x_0}(y) = f(g(Ty,Ty))$ for each $y \in X^*$, where $\mu_{x_0}(dx) := \mu(-x_0 + dx)$, $x \in X$. Since $\mathbf{K}$ is locally compact, then X^* is nontrivial and separates points of X (see [NB85, Roo78]). Each one-dimensional over $\mathbf{K}$ projection of a Gaussian measure is a Gaussian measure and products of Gaussian measures are Gaussian measures, hence convolutions of Gaussian measures are also Gaussian measures. Therefore, $\hat{\mu}_{x_0} : X^* \to \mathbf{C}$ is a nontrivial character: $\hat{\mu}_{x_0}(y_1 + y_2) = \hat{\mu}_{x_0}(y_1)\hat{\mu}_{x_0}(y_2)$ for each y_1 and y_2 in X^*. If $char(\mathbf{K}) = 0$ and $\mathbf{K}$ is a non-Archimedean field, then there exists a prime number p such that $\mathbf{Q_p}$ is the subfield of $\mathbf{K}$. Then $\hat{\mu}(p^n y) = (\hat{\mu}(y))^{p^n}$ for each $n \in \mathbf{Z}$ and $y \in X^* \setminus \{0\}$, particularly, for $n \in \mathbf{N}$ tending to the infinity we have $\lim_{n\to\infty} p^n y = 0$ and $\lim_{n\to\infty} \hat{\mu}_{x_0}(p^n y) = 1$, $\lim_{n\to\infty}(\hat{\mu}_{x_0}(y))^{p^n} = 0$, since $\lim_{n\to\infty} \hat{\mu}_{x_0}(p^{-n}y) = 0$ and $|\hat{\mu}_{x_0}(y)| < 1$ for $y \neq 0$. This gives the contradiction, hence $\mathbf{K}$ can not be a non-Archimedean field of zero characteristic. Suppose that $\mathbf{K}$ is a non-Archimedean field of characteristic $char(\mathbf{K}) = p > 0$, then $\mathbf{K}$ is isomorphic with the field of formal power series in variable t over a finite field $\mathbf{F_p}$. Therefore, $\hat{\mu}_{x_0}(py) = 1$, but $\hat{\mu}_{x_0}(y)^p \neq 1$ for $y \neq 0$, since $\lim_{n\to\infty} \hat{\mu}_{x_0}(t^{-n}y) = 0$. This contradicts the fact that $\hat{\mu}_{x_0}$ need to be the nontrivial character, consequently, $\mathbf{K}$ can not be a non-Archimedean field of nonzero characteristic as well. It remains the classical case of X over $\mathbf{R}$ or $\mathbf{C}$, but the latter case reduces to X over $\mathbf{R}$ with the help of the isomorphism of $\mathbf{C}$ as the $\mathbf{R}$-linear space with $\mathbf{R^2}$.

23. Theorem. *Let $\mu_{q,B,\gamma}$ and $\mu_{q,B,\delta}$ be two q-Gaussian measures. Then $\mu_{q,B,\gamma}$ is equivalent to $\mu_{q,B,\delta}$ or $\mu_{q,B,\gamma} \perp \mu_{q,B,\delta}$ according to $v_q(\gamma-\delta) \in B^{1/2}(\mathsf{D}_{B,Y})$ or not. The measure $\mu_{q,B,\gamma}$ is orthogonal to $\mu_{g,B,\delta}$, when $q \neq g$. Two measures $\mu_{q,B,\gamma}$ and $\mu_{g,A,\delta}$ are either equivalent or orthogonal.*

24. Theorem. *The measures $\mu_{q,B,\gamma}$ and $\mu_{q,A,\gamma}$ are equivalent if and only if there exists a positive definite bounded invertible operator T such that $A = B^{1/2}TB^{1/2}$ and $T - I \in L_2(Y^*)$.*

Proof. Using the projective limit reduce consideration to the Banach space X. Then proof of theorems $23, 24$ follows from the consideration of characteristic functionals of measures, the Kakutani theorem 3.3.1 and the fact that the Fourier transform F is the unitary operator on $L^2(\mathbf{K}, Bf(\mathbf{K}), v, \mathbf{C})$ due to the Parceval-Steklov equality, where v denotes the Haar normalized nonnegative measure on $\mathbf{K}$. Therefore, it is possible to proceed with the characteristic functionals $\hat{\mu}_{q,B,\delta}$ and $\hat{\mu}_{g,A,\gamma}$ instead of measures. If $g \neq q$ then the measure $\mu_{q,B,\gamma}$ is orthogonal to $\mu_{g,B,\delta}$, since $\lim_{R>0,R+n\to\infty}(\mu_{q,B,\gamma})_{X_n}(X^c_{R,n})/(\mu_{g,B,\delta})_{X_n}(X^c_{R,n}) = 0$, for each $q > g$, where $X^c_{R,n} := X_n \setminus B(X_n, 0, R)$, $X_n := span_{\mathbf{K}}(e_m : m = n, n+1, \dots, 2n)$, $(\mu_{q,B,\gamma})_{X_n}$ is the projection of the measure $\mu_{q,B,\gamma}$ on X_n. Each Hellinger integral in the Kakutani theorem is in $[0,1] \subset \mathbf{R}$, consequently, the product in the Kakutani theorem is either converging to a positive constant or diverges to zero, hence two measures $\mu_{q,B,\gamma}$ and $\mu_{g,A,\delta}$ are either equivalent or orthogonal (see also the classical case in § II.3 [Kuo75]).

1.7. Comments

Real-valued Haar measures on locally compact topological groups are described in details, for example, in [Bou63-69, HR79, FD88].

S.N. Evans had used stochastic approach to study non-Archimedean analogs of Gaussian measures [Eva89, Eva91].

1. Definition. Consider a normed space $(Y, \|*\|_Y)$ over a non-Archimedean field $\mathbf{F}$. A subset X in Y is called orthogonal if for each finite subset $\{x_1,\dots,x_n\} \subset X$ and each $a_1,\dots,a_n$ is satisfied the equality: $\|\sum_{j=1}^n a_j x_j\|_Y = \max_{1\le j\le n} \|a_j\| \|x_j\|_Y$. An orthogonal subset X is called orthonormal, if $\|x\|_Y = 1$ for each $x \in X$.

2. Definition. Let $(E, \mathcal{E})$ be a measurable vector space over a local field $\mathbf{F}$. A random variable $\xi \in L^2(E,P,\mathbf{R})$ is called a Gaussian random variable (is distributed by a Gaussian probability measure) if for each two independent copies ξ_1 and ξ_2 of ξ and each orthonormal vectors (a_1,a_2) and $(b_1,b_2) \in \mathbf{F}^2$ the pair (ξ_1,ξ_2) has the same distribution as $(a_1\xi_1 + a_2\xi_2, b_1\xi_1 + b_2\xi_2)$.

3. Theorem. *A random variable $\xi \in L^2(E,P,\mathbf{R})$ that is not almost surely zero is Gaussian if and only if its distribution is a cutoff of the Haar measure λ: $P(\{\xi \in dx\}) = \pi^{-n}\psi(\pi^{-n}\|x\|)\lambda(dx)$ for some $n \in \mathbf{Z}$, where ψ is the characteristic function of $[0,1] \subset \mathbf{R}$, $|\pi| < 1$ is the generator of the normalization group $\Gamma_{\mathbf{F}}$, E is locally compact.*

4. Theorem. *Suppose $(E, \|*\|_E)$ is a separable Banach space with topological dual space E^* and P is a $\mathbf{F}$-Gaussian measure on E. Put $S := \{x \in E : |T(x)| \le \|T(X)\|_\infty \quad \forall T \in E^*\}$. Then*

(i) the group S is the closed support of P;

(ii) the group S is compact;

(iii) if M is a measurable vector subspace of E, then $P(M)$ is either 1 or 0, depending on whether $S \subset M$ or not.

5. Remark. A random variable $\xi \in L^2(E,P,\mathbf{R})$ was called also a random variable with values in E in terminology of S.N. Evans (see also [DF91, Eva88, Eva89, Eva91]). Gaussian random fields with values in $\mathbf{Q_p}$ and controlled by real-valued measures were also constructed in [AK91] with the help of forward and backward Kolmogorov equations. The property of invariance of the Gaussian distribution under orthogonal non-Archimedean transformations leads to a measure with compact support and equivalent to the Haar measure up to a constant multiplier. Together with results above this gives another proof of the fact that in the non-Archimedean case there does not exist a measure having all the same properties as the Gaussian measure in a real Banach space. On the other hand, it shows that the orthogonality condition being too strong in the non-Archimedean case leads to a measure, which is a restriction of the Haar measure on a compact subset in the case of locally compact E. Theorem 4 shows, that S.N. Evans has considered measures with compact support also in a non-locally compact $\mathbf{F}$-linear space E. This means, that such measures of Theorems $3,4$ are not quasi-invariant, but in the classical case Gaussian measures are quasi-invariant.

V.S. Vladimirov (see [Vla89, VVZ94] and references therein) and A.N. Kochubei [Koc95] also have considered integrals over $\mathbf{Q_p}$ of the function $f(x) := \chi_p(ax^2+bx)$ with parameters $a,b \in \mathbf{Q_p}$ as analogs of Gaussian integrals and they evaluated them, where χ_p is the complex valued character of $\mathbf{Q_p}$ as the additive group. Certainly, because of $|\chi_p(z)| = 1$ for each $z \in \mathbf{Q_p}$, the function $\chi_p(ax^2+bx)$ has not characteristic graph of the Gaussian distribution tending to zero while $|x|$ tends to the infinity.

All this terminology Gaussian integrals and Gaussian measures is very conditional and optional, since in the times of K.F. Gauss no any non-Archimedean fields were studied and non-Archimedean analysis was not existent, because non-Archimedean fields had begun to be investigated only in the end of the 19-th century and the non-Archimedean normalization comes back from the Ostrowski's theorem (see [Roo78, Wei73]). Any use of suitable non-Archimedean measures or integrals depends on concrete problems.

Pseudo-differential operators considered in this chapter were first studied by V.S. Vladimirov [Vla89] and also were used in non-Archimedean quantum mechanics [VV89, VVZ94]. They were used for solutions of the non-Archimedean analog of the heat equation (see, for example, [Koc96]). It was proved [VVZ94], for example, for $p \neq 2$ and $a \neq 0$, that $\int_{B(\mathbf{Q_p},0,p^N)} \chi_p(ax^2+bx)dx =: I(p,N,a,b)$ has values $I(p,N,a,b) = p^N \omega(p^N|b|_p)$, when $|a|_p p^{2N} \leq 1$,

$$I(p,N,a,b) = \lambda_p(a)|a|_p^{-1/2}\chi_p(-b^2/(4a))\omega(p^{-N}|b/a|_p),$$

when $|a|_p p^{2N} > 1$, where $\lambda_p(a) = 1$ for even ν, $\lambda_p(a) = \left(\frac{a_0}{p}\right)$, if ν is odd and $p = 1 \pmod 4$, $\lambda_p(a) = i\left(\frac{a_0}{p}\right)$, if ν is odd and $p = 3 \pmod 4$, where ν is such that $a = p^\nu(a_0 + a_1 p + \cdots)$, $\nu \in \mathbf{Z}$, $a_j \in \{0,1,\dots,p-1\}$, $a \in \mathbf{Q_p}$, $a_0 \neq 0$, $\left(\frac{n}{p}\right)$ is the Legendre symbol for each prime p and n an integer prime to p, such that $\left(\frac{n}{p}\right) = 1$ for n being a quadratic residue modulo p, and $\left(\frac{n}{p}\right) = -1$ otherwise, $\omega(y) = 1$ for $0 \leq y \leq 1$, $\omega(y) = 0$ for $y > 1$, dx denotes the Haar nonnegative measure on $\mathbf{Q_p}$. Taking the limit while N tends to infinity gives

$$\int_{\mathbf{Q_p}} \chi_p(ax^2+bx)dx := \lim_{N\to\infty} I(p,N,a,b) = \lambda_p(a)|a|_p^{-1/2}\chi_p(-b^2/(4a)).$$

For a local field $\mathbf{K}$ and the character χ of rank zero of $\mathbf{K}$ as the additive group and a radial function $f(x) := g(\|x\|)$ it was also evaluated the integral

$$\int_{\mathbf{K}} \chi(x\xi)f(x)dx = (1-q^{-1})\|\xi\|^{-1}\sum_{n=0}^{\infty} q^{-n}g(q^{-n}\|\xi\|^{-1}) - \|\xi\|^{-1}g(q\|\xi\|^{-1})$$

for each $\xi \neq 0$, where $\mathbf{K}$ is the finite algebraic extension of $\mathbf{Q_p}$ and $q := p^f$, $f \geq 1$ is the index of inertia, $ef = (\mathbf{K} : \mathbf{Q_p})$, $e \geq 1$ is the ramification index, $B(\mathbf{K},0,1)/\{x \in \mathbf{K} : |x| < 1\} = \mathbf{F}_q$ is the finite field consisting of q elements.

There is a generalization of the Minlos theorem on locally $\mathbf{K}$-convex spaces. In the case of $\mathbf{K} = \mathbf{R}$ it states, that there exists a bijective correspondence between continuous positive definite functions on a nuclear locally convex topological vector space and Radon probability measures on the Borel σ-algebra of the topological conjugate space E^* supplied with the weak topology $\sigma(E^*,E)$ provided by the Fourier transform. For properties of measures to be quasi-invariant or pseudo-differentiable it does not play any role. This theorem only permits to get an information about a measure by its characteristic functional, that is also very important. Mądrecki has proved its non-Archimedean analog in [Mad91c].

6. Theorem. *Let E be a Hausdorff locally* **K***-convex space, where* **K** *is a local field. Then any continuous positive definite function on E is the Fourier transform of a Radon probability measure on the Borel* σ*-field $Bf(E^*)$ of the topological conjugate space E^* supplied with the $*$-weak topology $\sigma(E^*,E)$. Conversely, if E is barreled, then the Fourier transform of a Radon probability measure on $Bf(E^*)$ is a continuous positive definite function on E.*

7. Remark. In [Sat94] an analog of a Wiener measure in a Banach space with a measurable norm was studied by T. Satoh. It was further development of some results of S.N. Evans.

8. Definitions. A countable subset $\{e_i : i \in \mathbf{N}\}$ is called an orthogonal Schauder base in a normed space H over a non-Archimedean field **K** if it satisfies the following two conditions:

(1) for each $v \in H$ there is the unique sequence $\{c_i : c_i \in \mathbf{K}, i \in \mathbf{N}\}$ such that $v = \sum_i c_i e_i$;

(2) for any converging series $\sum_i c_i e_i$, we have $|\sum_{i=1}^{\infty} c_i e_i| = \max_i |c_i e_i|$.

If in addition to the above conditions $|e_i| = 1$ for each i, then $\{e_i : i\}$ is called an orthonormal Schauder basis.

A **K**-linear map $P \in L(H)$ is called an orthogonal projection, if $P^2 = P$ and $Im(P) := P(H)$ is orthogonal to $Ker(P) := P^{-1}(0)$.

A probability real-valued measure ν on **K** is said to be admissible, if:

(1) the measure ν is isometry invariant absolute continuous with respect to the Haar measure μ and

(2) the value of the Radon-Nykodim derivative $d\nu/d\mu(x)$ at $x = \pi^n$ is a non-decreasing function of n, where $|\pi| < 1$ is generator of the normalization group $\Gamma_{\mathbf{K}}$.

The cylinder measure G_ν with parameter ν is the function on $Cyl(H)$ defined by the following formula: $G_\nu(P^{-1}(F)) := \nu^{P(H)}(F)$, where $Cyl(H)$ denotes the algebra of all cylinder subsets defined with the help of orthogonal projections on finite dimensional over **K** subspaces in H, $dim_{\mathbf{K}} P(H) < \infty$, $P^{-1}(F) \in Cyl(H)$, $F \in Bf(P(H))$.

A semi-norm $\|*\|$ in H is called measurable if for every $\varepsilon > 0$ there exists $P \in FOP(H)$ satisfying $\|x\| \le \varepsilon\|x\|$ for each $x \in Ker(P)$, where $FOP(H)$ denotes the family of all orthogonal projection operators with finite-dimensional ranges over **K**.

Suppose B is a completion of H relative to the measurable norm $\|*\|$ in H and B^* be the topological dual space of all continuous **K**-linear functionals $f : B \to \mathbf{K}$.

9. Proposition. *The measure G_ν is not* σ*-additive.*

10. Definitions. Let $Cyl^*(B)$ be the family of all cylinder subsets in B of the form: $T := \{x \in B : (P_1(x), \dots, P_n(x)) \in E\}$, where $E \in Af(\mathbf{K}^n, \mu^n)$, $P_1, \dots, P_n \in Cyl(H)$.

The Wiener measure W_ν with parameter ν is defined on $Cyl^*(B)$ of the form: $W_\nu(T) := G_\nu(T \cap H)$ for each $T \in Cyl^*(H)$.

11. Theorem. *The Wiener measure W_ν extends to a* σ*-additive measure on the sigma algebra $\sigma(Cyl^*(B))$ generated by $Cyl^*(B)$.*

12. Remark. Relations between topologies and Borel structures are given, for example, in [Chr74, Kur66] and references therein.

13. Definitions. By a Borel structure $\mathcal{B}(X)$ on a set X there is understood a system (a particular subfamily) of subsets of X which is closed with respect to the operations of complement and countable union, and which is nonempty. Thus $\mathcal{B}(X)$ is the paving of X, it is also called a σ-field.

A (Borel) measurable mapping f of $(X,\mathcal{B}(X))$ into $(Y,\mathcal{B}(Y))$ is a mapping $f:X\to Y$ such that $f^{-1}(B)\in\mathcal{B}(X)$ for each $B\in\mathcal{B}(Y)$. A Borel isomorphism is a bijective mapping f of X onto Y such that both f and f^{-1} are (Borel) measurable.

A measurable space $(X,\mathcal{B}(X))$ is called separated, if for all $x\neq y\in X$ there exists $A\in\mathcal{B}(X)$ such that $x\in A$, but $y\notin A$.

A measurable space $(X,\mathcal{B}(X))$ is called separable, if there exists a sequence $\{A_n\in\mathcal{B}(X): n\in\mathbf{N}\}$ which generates $\mathcal{B}(X)$. It is called countably separated, if there exists a separable subfield which is separated.

14. Theorem. *A measurable space $(X,\mathcal{B}(X))$ is Borel isomorphic with a subset of the real segment $[0,1]$ equipped with the subspace Borel structure (induced from the Borel structure generated by the usual topology on $[0,1]$) if and only if it is separable and separated.*

15. Definitions. A Polish space X is a Hausdorff topological space which can be equipped with a metric d generating its topology and (X,d) is complete.

An analytic topological space Y is a Hausdorff space which is the continuous image of a Polish space X.

If the Hausdorff topological space Y is an injective continuous image of a Polish space, it is called a standard topological space.

16. Theorem. *If X is an analytic topological space, then there exists a surjective continuous mapping f from $\mathbf{N}^{\aleph_0}$ onto X, where $\mathbf{N}^{\aleph_0}$ is supplied with the product topology.*

17. Definitions. Let (X,τ_X) be a topological space with a Hausdorff topology τ_Y, then $A\subset X$ is called of the first category (in X), if A is a countable union of closed nowhere dense subsets in X. Subsets A of X which can not be so represented are called of the second category. The BP-field is the σ-field of subsets A of X for which there exists $U\in\tau_X$ such that $A\triangle U:=(A\setminus U)\cup(U\setminus A)$ is of the first category. A Hausdorff topological space is called a Baire space, if every its open subset is of the second category.

A topological group G is called analytic, if G is analytic as a topological space. A topological group G is called σ-bounded, if G can be covered with countably many left translations of every neighborhood.

18. Theorem. *Let G be a topological group and A be a BP-measurable subset of the second category. Then $A\circ A^{-1}$ is a neighborhood.*

19. Theorem. *Any BP-measurable homomorphism f from a topological group G which is of the second category in itself to a σ-bounded topological group H is continuous. If both G and H are analytic and G is of the second category, then each homomorphism $f:G\to H$ with analytic graph is continuous.*

20. Theorem. *Let (G,τ_G) be a Baire topological group with a quasi-invariant nontrivial nonnegative measure μ relative to a dense Baire subgroup $(G',\tau_{G'})$ (in itself) with a BP-measurable quasi-invariance factor $\rho_\mu(h,g): G'\times G\to\mathbf{R}$, then $\rho_\mu(h,g)$ is continuous in $(h,g)\in G'\times G$.*

Proof. It follows from Theorems 18, 19 using the co-cycle property of a quasi-invariance factor $\rho_\mu(hv,g)=\rho_\mu(v,h^{-1}g)\rho_\mu(h,g)$ on a topological group G relative to the left action of a dense subgroup G', where $\rho_\mu(h,g):=\mu(h^{-1}dg)/\mu(dg)$, μ is a measure on G. Let B be a Borel subset in $\mathbf{R}$, then $\rho_\mu^{-1}(B)=:A$ is a BP-measurable subset in $(G'\times G,\tau_{G'}\times\tau_G)$.

In particular, take B open in $\mathbf{R}$. In view of the co-cycle condition and Definition 17 there exists V of the second category in $G' \times G$ such that V is BP-measurable and $A \triangle V$ is of the first category. Since $(G' \times G, \tau_{G'} \times \tau_G)$ is the Baire topological group, then $A \circ A^{-1}$ is a neighborhood of the unit element in $G' \times G$.

21. Note. The latter theorem shows, that the supposition of continuity of the quasi-invariance factor of this chapter is not very restrictive. Moreover, it is implied also under milder conditions. In details relations between Borel measurability and continuity of functions $\phi : (G' \times X) \ni (h,x) \mapsto hx := \phi(h,x) \in X$ satisfying conditions $\phi(e,x) = x$, $\phi(v, \phi(h,x)) = \phi(vh,x)$ for each $v,h \in G'$ and each $x \in X$ were given in [Fid00], where X is a Polish topological space and G' is a Polish topological group.

Chapter 2

Non-Archimedean Valued Measures

2.1. Introduction

This Chapter is the continuation of the first one and treats the case of measures with values in non-Archimedean fields of zero characteristic, for example, the field $\mathbf{Q_p}$ of p-adic numbers. There are specific features with formulations of definitions and theorems and their proofs, because of differences in the notions of σ-additivity of real-valued and $\mathbf{Q_p}$-valued measures, differences in the notions of spaces of integrable functions, quasi-invariance and pseudo-differentiability. For the s-free group G a measure m with values in a non-Archimedean field $\mathbf{K_s}$ satisfy Condition $I.1.(H)$ only for an algebra of clopen (closed and open) subsets A, where a field $\mathbf{K_s}$ is a finite algebraic extension of $\mathbf{Q_s}$. Indeed, in the last case if a measure is locally finite and σ-additive on the Borel algebra of G, then it is purely atomic with atoms being singletons, so it can not be invariant relative to the entire Borel algebra (see Chapters 7-9 [Roo78]). The Lebesque convergence theorem has quite another meaning, the Radon-Nikodym theorem in its classical form is not applicable to the considered here case. A lot of definitions and theorems given below are the non-Archimedean analogs of results for real-valued measures of Chapter I. Frequently their formulations and proofs differ strongly. If proofs differ slightly from the case of real-valued measures of Chapter I, only general circumstances are given in for non-Archimedean-valued measures.

In § 2 sequences of weak distributions, characteristic functions of measures and their properties are defined and investigated. The non-Archimedean analogs of the Minlos-Sazonov and Bochner-Kolmogorov theorems are given. Quasi-measures also are considered. In § 3 products of measures are considered together with their density functions. The non-Archimedean analog of the Kakutani theorem is investigated. In the present chapter broad classes of quasi-invariant measures are defined and constructed. Theorems about quasi-invariance of measures under definite linear and non-linear transformations $U : X \to X$ are proved. § 4 contains a notion of pseudo-differentiability of measures. This is necessary, because for functions $f : \mathbf{K} \to \mathbf{Q_s}$ with $s \neq p$ there is not any notion of differentiability (there is not such non-linear non-trivial f), where $\mathbf{K}$ is a field such that $\mathbf{K} \supset \mathbf{Q_p}$. There are given criteria for the pseudo-differentiability. In § 5 there are given theorems about convergence of measures with taking into account their quasi-invariance and pseudo-differentiability, that is, in the corresponding spaces of measures. The main results are Theorems 2.21, 2.30, 3.5, 3.6, 3.15, 3.19, 3.20, 4.2, 4.3, 4.5, 5.7-5.10.

In this chapter notations of Chapter I are used also.

Notations. Henceforth, $\mathbf{K}$ denotes a locally compact infinite field with a non-trivial norm, then the Banach space X is over $\mathbf{K}$. In the present chapter measures on X have values in the field $\mathbf{K_s}$, where $\mathbf{K_s}$ is a non-Archimedean field complete relative to its uniformity and such that $\mathbf{Q_s} \subset \mathbf{K_s}$, where $\mathbf{Q_s}$ is the s-adic field with the certain prime number s. In all theorems of this chapter and Chapter IV measures can be realized, when $\mathbf{K_s}$ is a local field, that is, a finite algebraic extension of $\mathbf{Q_s}$, but also they are true, when $\mathbf{K_s}$ is a broader field, for example, $\mathbf{C_s} \subset \mathbf{K_s}$ or $\mathbf{U_s} \subset \mathbf{K_s}$ (see below). Henceforth, $\mathbf{C_s}$ denotes the uniform completion of the union of all local fields $\mathbf{K_s}$ with the multiplicative ultra-norm extending that of $\mathbf{Q_s}$. Let $\mathbf{U_s}$ be a field obtained from $\mathbf{C_s}$ with the help of procedures of ultra-products and spherical completion such that its normalization group $\Gamma_{\mathbf{U_s}} = (0,\infty)$ (see [Dia84, Esc95] and references therein and comments below). We assume that $\mathbf{K}$ is s-free as the additive group, for example, either $char(\mathbf{K}) = 0$, $\mathbf{K}$ is a finite algebraic extension of the field of p-adic numbers $\mathbf{Q_p}$ or $char(\mathbf{K}) = p$ and $\mathbf{K}$ is isomorphic with a field $\mathbf{F_p}(\theta)$ of formal power series consisting of elements $x = \sum_j a_j\theta^j$, where $a_j \in \mathbf{F_p}$, $|\theta| = p^{-1}$, $\mathbf{F_p}$ is a finite field of p elements, p is a prime number and $p \neq s$. These imply that $\mathbf{K}$ has the Haar measures with values in $\mathbf{K_s}$ [Roo78]. If X is a Hausdorff topological space with a small inductive dimension $ind(X) = 0$, then E denotes an algebra of subsets of X, as a rule $E \supset Bco(X)$ for $\mathbf{K_s}$-valued measures, where $Bco(X)$ denotes an algebra of clopen (closed and open) subsets of X, $Bf(X)$ is a Borel σ-field of X, $Af(X,\mu)$ is the completion of E by a measure μ in § 2.1.

2.2. Non-Archimedean Valued Distributions

2.1. For a Hausdorff topological space X with a small inductive dimension $ind(X) = 0$ [Eng86], henceforth, measures μ are given on a measurable space (X,E), where E is an algebra such that $E \supset Bco(X)$, $Bco(X)$ is an algebra of closed and at the same time open (clopen) subsets in X.

We recall that a mapping $\mu : E \to \mathbf{K_s}$ for an algebra E of subsets of X is called a measure, if the following conditions are accomplished:

$$(i)\ \mu \text{ is additive and } \mu(\emptyset) = 0,$$

$$(ii) \text{ for each } A \in E \text{ there exists the following norm}$$

$\|A\|_\mu := \sup\{|\mu(B)|_{\mathbf{K_s}} :\ B \subset A, B \in E\} < \infty,$

$$(iii) \text{ if there is a shrinking family } F, \text{ that is, for each}$$

$A, B \in F$ there exist $F \ni C \subset (A \cap B)$ and $\cap\{A : A \in F\} = \emptyset$, then $\lim_{A\in F} \mu(A) = 0$ (see also Chapter 7 [Roo78] and also about the completion $Af(X,\mu)$ of the algebra E by the measure μ). A measure with values in $\mathbf{K_s}$ is called a probability measure if $\|X\|_\mu = 1$ and $\mu(X) = 1$. For functions $f : X \to \mathbf{K_s}$ and $\phi : X \to [0,\infty)$ there are used notations $\|f\|_\phi := \sup_{x\in X}(|f(x)|\phi(x))$, $N_\mu(x) := \inf(\|U\|_\mu :\ U \in Bco(X),\ x \in X)$. Tight measures (that is, measures defined on $E \supset Bco(X)$) compose the Banach space $\mathsf{M}(X)$ with a norm $\|\mu\| := \|X\|_\mu$. Everywhere below there are considered measures with $\|X\|_\mu < \infty$ for μ with values in $\mathbf{K_s}$, if it is not specified another.

A measure μ on E is called Radon, if for each $\varepsilon > 0$ there exists a compact subset $C \subset X$ such that $\|\mu|_{(X\setminus C)}\| < \varepsilon$. Henceforth, $\mathsf{M}(X)$ denotes the space of norm-bounded measures, $\mathsf{M}_t(X)$ is its subspace of Radon norm-bounded measures.

2.1.1. Definition. Suppose that $\mathcal{S}$ is a subfamily of a covering ring $\mathcal{R}$ of X such that for each A and B in $\mathcal{S}$ there exists $C \in \mathcal{S}$ with $C \subset A \cap B$, then $\mathcal{S}$ is called shrinking. For a function $f : \mathcal{R} \to \mathbf{K}$ or $f : \mathcal{R} \to \mathbf{R}$ the notation $\lim_{A\in\mathcal{S}} f(A) = 0$ means that for each $\varepsilon > 0$ there exists $B \in \mathcal{S}$ such that $|f(A)| \le \varepsilon$ for each $A \in \mathcal{S}$ with $A \subset B$.

2.1.2. Notes. Put $\|f\|_\mu := \|f\|_{N_\mu}$. Then for each $A \subset X$ the function $\|A\|_\mu := \sup_{x\in A} N_\mu(x)$ is defined such that its restriction on $\mathcal{R}$ coincides with that of given by Equation 2.1.(ii) (see also Chapter 7 [Roo78]). A $\mathcal{R}$-step function f is a function $f : X \to \mathbf{K}$ such that it is a finite linear combination over $\mathbf{K}$ of characteristic functions Ch_U of $U \in \mathcal{R}$. A function f is called μ-integrable if there exists a sequence $\{f_n : n \in \mathbf{N}\}$ of step functions such that $\lim_{n\to\infty} \|f - f_n\|_{N_\mu} = 0$. The Banach space of μ-integrable functions is denoted by $L(\mu) := L(X, \mathcal{R}, \mu, \mathbf{K})$. There exists a ring $\mathcal{R}_\mu$ of subsets A in X for which $Ch_A \in L(\mu)$. The ring $\mathcal{R}_\mu$ is the extension of the ring $\mathcal{R}$ such that $\mathcal{R}_\mu \supset \mathcal{R}$.

For example, if $\mathbf{K}$ is locally compact, then the normalization group $\Gamma_{\mathbf{K}} := \{|x| : x \in \mathbf{K}, x \ne 0\}$ is discrete in $(0,\infty) \subset \mathbf{R}$. If μ is a measure such that $0 < \|\mu\| < \infty$, then there exists $a \in \mathbf{K}$ such that $|a| = \|\mu\|^{-1}$, since $\|\mu\| \in \Gamma_{\mathbf{K}}$ for discrete $\Gamma_{\mathbf{K}}$, hence $a\mu$ is also the measure with $\|\mu\| = 1$. If $\|\mu\| = 1$, then μ is the nonzero measure. For such μ with $\mu(X) =: b_X \in \mathbf{K}$ if $b_X \ne 1$ we can take a non-void new set Y and define on $X_0 := Y \cup X$ a minimal ring $\mathcal{R}_0$ generated by $\mathcal{R}$ and $\{Y\}$, that is, $\mathcal{R}_0 \cap Y = \{\emptyset, \{Y\}\}$ and $\mathcal{R}_0 \supset \mathcal{R} \cup \{Y\}$. Since $\|\mu\| = 1$, then $|b_X| \le 1$. Put $\mu(Y) := 1 - b_X$, then there exists the extension of μ from $\mathcal{R}$ on $\mathcal{R}_0$ such that $\|\mu\| = 1$ and $\mu(X_0) = 1$, since $|1 - b_X| \le \max(1, |b_X|) = 1$. In particular, we can take a singleton $Y = \{y\}$. Therefore, probability measures are rather naturally related with nonzero bounded measures. This also shows that from $\|\mu\| = 1$ in general does not follow $\mu(X) = 1$. Evidently, from $\mu(X) = 1$ in general does not follow $\|\mu\| = 1$, for example, $X = \{0,1\}$, $\mathcal{R} = \{\emptyset, \{0\}, \{1\}, X\}$, $\mu(\{0\}) = a$, $\mu(\{1\}) = 1 - a$, where $|a| > 1$, hence $\|\mu\| = |a| > 1$.

Consider a non-void topological space X. A topological space is called zero-dimensional if it has a base of its topology consisting of clopen subsets. A topological space X is called a T_0-space if for each two distinct points x and y in X there exists an open subset U in X such that either $x \in U$ and $y \in X \setminus U$ or $y \in U$ and $x \in X \setminus U$.

A covering ring $\mathcal{R}$ of a space X defines on it a base of zero-dimensional topology $\tau_{\mathcal{R}}$ such that each element of $\mathcal{R}$ is considered as a clopen subset in X. If $\pi : X \to Y$ is a mapping such that $\pi^{-1}(\mathcal{R}_Y) \subset \mathcal{R}_X$, then a measure μ on $(X, \mathcal{R}_X)$ induces a measure $\nu := \pi(\mu)$ on $(Y, \mathcal{R}_Y)$ such that $\nu(A) = \mu(\pi^{-1}(A))$ for each $A \in \mathcal{R}_Y$.

2.2. If $A \in Bco(L)$, then $P_L^{-1}(A)$ is called a cylindrical subset in X with a base A, $B^L := P_L^{-1}(Bco(L))$, $B_0 := \cup(B^L : L \subset X, L$ is a Banach subspace , $dim_{\mathbf{K}} X < \aleph_0)$ (see §I.2.2). Let an increasing sequence of Banach subspaces $L_n \subset L_{n+1} \subset \cdots$ such that $cl(\cup[L_n : n]) = X$, $dim_{\mathbf{K}} L_n = \kappa_n$ for each n be chosen, where $cl(A) = \bar{A}$ denotes a closure of A in X for $A \subset X$. We fix a family of projections $P_{L_n}^{L_m} : L_m \to L_n$ such that $P_{L_n}^{L_m} P_{L_k}^{L_n} = P_{L_k}^{L_m}$ for each $m \ge n \ge k$. A projection of the measure μ onto L denoted by $\mu_L(A) := \mu(P_L^{-1}(A))$ for each $A \in Bco(L)$ compose the consistent family:

$$\mu_{L_n}(A) = \mu_{L_m}(P_{L_n}^{-1}(A) \cap L_m) \tag{1}$$

for each $m \ge n$, since there are projectors $P_{L_n}^{L_m}$, where $\kappa_n \le \aleph_0$ and there may be chosen

$\kappa_n < \aleph_0$ for each n.

An arbitrary family of measures $\{\mu_{L_n} : n \in \mathbf{N}\}$ having property (1) is called a sequence of weak distributions (see also [DF91, Sko74]).

2.3. Lemma. *A sequence of weak distributions $\{\mu_{L_n} : n\}$ is generated by some measure μ on $Bco(X)$ if and only if for each $c > 0$ there exists $b > 0$ such that $\|L_n \setminus B(X,0,r)\|_{\mu_{L_n}} \le c$ for each $n \in \mathbf{N}$ and $\sup_n \|L_n\|_{\mu_{L_n}} < \infty$ for μ with values in $\mathbf{K_s}$, where $r \ge b$.*

Proof. For μ with values in $\mathbf{K_s}$ the necessity is evident.

Recall Theorem 7.6 [Roo78]:

(Ri). If μ is a measure on $\mathcal{R}$, then N_μ is $\mathcal{R}$-upper semi-continuous and for each $A \in \mathcal{R}_\mu$ and $b > 0$ the set $\{x \in A : N_\mu(x) \ge b\}$ is $\mathcal{R}_\mu$-compact (hence $\mathcal{R}$-compact);

(Rii). Conversely, let $\mu : \mathcal{R} \to \mathbf{K}$ be additive. Assume that there exists an $\mathcal{R}$-upper semi-continuous function $\phi : X \to [0,\infty)$ such that $|\mu(A)| \le \sup_{x\in A} \phi(x)$ for each $A \in \mathcal{R}$ and $\{x \in A : \phi(x) \ge b\}$ is compact for every $b > 0$. Then μ is a measure and $N_\mu \le \phi$.

To prove the sufficiency it remains only to verify property (2.1.iii), since then $\|X\|_\mu = \sup_n \|L_n\|_{\mu_{L_n}} < \infty$. Let $B(n) \in E(L_n)$, $A(n) = P_{L_n}^{-1}(B(n))$, by the cited above Theorem 7.6 [Roo78] for each $c > 0$ there is a compact subset $C(n) \subset B(n)$ such that $\|B(n) \setminus C(n)\|_{\mu_{L_n}} < c$, where $\|B(n) \setminus D(n)\|_\mu \le \max(\|B(m) \setminus C(m)\|_{\mu_{L(m)}} : m = 1,\dots,n) < c$ and $D(n) := \bigcap_{m=1}^n P_{L(m)}{}^{-1}(C(m)) \cap L_n)$, $P_{L_n}^{-1}(E(L_n) \subset E = E(X)$. If $A(n) \supset A(n+1) \supset \cdots$ and $\bigcap_n A(n) = \emptyset$, then $A'(n+1) \subset A'(n)$ and $\bigcap_n A'(n) = \emptyset$, where $A'(n) := P_{L_n}^{-1}(D(n))$, hence $\|A(n)\|_\mu \le \|A'(n)\|_\mu + c$. There may be taken $B(n)$ as closed subsets in X.

Remind the Hahn-Banach Theorem 4.8 [Roo78]: let E and F be normed spaces, D a linear subspace of E; assume that either D or F is spherically complete; then every $S \in L(D,F)$ has an extension $\overline{S} \in L(E,F)$ such that $\|\overline{S}\| = \|S\|$.

In accordance with the Alaouglu-Bourbaki theorem (see Exer. 9.202(a.3) [NB85]) if $\mathbf{K}$ is a locally compact field with a non-Archimedean multiplicative norm, X is a locally $\mathbf{K}$-convex space and U is a neighborhood of zero in X, then its polar $U^o := \{f \in X' : \sup_{x\in U} |f(x)| \le 1\}$ is $\sigma(X',X)$-compact.

In view of the Alaoglu-Bourbaki theorem and the Hahn-Banach theorem sets $A(n)$ and $B(X,0,r)$ are weakly compact in X, hence, for each $r > 0$ there exists n with $B(X,0,r) \cap A(n) = \emptyset$. Therefore, $\|A(n)\|_\mu = \|B(n)\|_{\mu_{L_n}} \le \|L_n \setminus B(X,0,r)\|_{\mu_{L_n}} \le c$ and there exists $\lim_{n\to\infty} \mu(A(n)) = 0$, since c is arbitrary.

2.4. Definition and notations. A function $\phi : X \to \mathbf{K_s}$ of the form $\phi(x) = \phi_S(P_S x)$ is called a cylindrical function if ϕ_S is a $E(S)$-measurable function on a finite-dimensional over $\mathbf{K}$ space S in X. For $\phi_S \in L(S,\mu_S,\mathbf{K_s}) := L(\mu_S)$ for μ with values in $\mathbf{K_s}$ we may define an integral by a sequence of weak distributions $\{\mu_{S(n)}\}$:

$$\int_X \phi(x)\mu_*(dx) := \int \phi_{S(n)}(x)\mu_{S(n)}(dx),$$

where $L(\mu)$ is the Banach space of classes of μ-integrable functions ($f = g$ μ-almost everywhere, that is, $\|A\|_\mu = 0$, $A := \{x : f(x) \ne g(x)\}$ is μ-negligible) with the following norm $\|f\| := \|g\|_{N_\mu}$.

2.5. Remarks and definitions. In the notation of § I.2.6 all continuous characters $\chi : \mathbf{K} \to \mathbf{C_s}$ have the form

$$\chi_\xi(x) = \varepsilon^{z^{-1}\eta((\xi,x))} \qquad (1)$$

for each $\eta((\xi,x)) \neq 0$, $\chi_\xi(x) := 1$ for $\eta((\xi,x)) = 0$, where $\varepsilon = 1^z$ is a root of unity, $z = p^{ord(\eta((\xi,x)))}$, $\pi_j : \mathbf{K} \to \mathbf{R}$, $\eta(x) := \{x\}_p$ and $\xi \in \mathbf{Q_p^{n}}^* = \mathbf{Q_p^n}$ for $char(\mathbf{K}) = 0$, $\eta(x) := \pi_{-1}(x)/p$ and $\xi \in \mathbf{K}^* = \mathbf{K}$ for $char(\mathbf{K}) = p > 0$, $x \in \mathbf{K}$, (see also § 25 [HR79]). Each χ is locally constant, hence $\chi : \mathbf{K} \to \mathbf{T_s}$ is also continuous, where $\mathbf{T}$ denotes the discrete group of all roots of 1, $\mathbf{T_s}$ denotes its subgroup of elements with orders that are not degrees s^m of s, $m \in \mathbf{N}$.

For a measure μ with values in $\mathbf{K_s}$ there exists a characteristic functional (that is, called the Fourier-Stieltjes transformation) $\theta = \theta_\mu : C(X,\mathbf{K}) \to \mathbf{U_s}$:

$$\theta(f) := \int_X \chi_e(f(x))\mu(dx), \tag{2}$$

where $\mathbf{K_s} \cup \mathbf{C_s} \subset \mathbf{U_s}$, either $e = (1,\dots,1) \in \mathbf{Q_p^n}$ for $char(\mathbf{K}) = 0$ or $e = 1 \in \mathbf{K}^*$ for $char(\mathbf{K}) = p > 0$, $x \in X$, f is in the space $C(X,\mathbf{K})$ of continuous functions from X into $\mathbf{K}$, in particular for $z = f$ in the topologically conjugated space X^* over $\mathbf{K}$, $z : X \to \mathbf{K}$, $z \in X^*$, $\theta(z) =: \hat{\mu}(z)$. It has the following properties:

$$\theta(0) = 1 \text{ for } \mu(X) = 1 \tag{3a}$$

and $\theta(f)$ is bounded on $C(X,\mathbf{K})$;

$$\sup_f |\theta(f)| = 1 \text{ for probability measures ;} \tag{3b}$$

$$\theta(z) \text{ is weakly continuous, that is, } (X^*,\sigma(X^*,X))\text{-continuous,} \tag{4}$$

$\sigma(X^*,X)$ denotes a weak topology on X^*, induced by the Banach space X over $\mathbf{K}$. To each $x \in X$ there corresponds a continuous linear functional $x^* : X^* \to \mathbf{K}$, $x^*(z) := z(x)$, moreover, $\theta(f)$ is uniformly continuous relative to the norm on

$$C_b(X,\mathbf{K}) := \{f \in C(X,\mathbf{K}) : \|f\| := \sup_{x\in X} |f(x)|_{\mathbf{K}} < \infty\}.$$

Recall the non-Archimedean analog of the Lebesgue theorem (see Exer. 7.F [Roo78]) for μ with values in $\mathbf{K_s}$. Let μ be a measure on $\mathcal{R}$, let $g \in L(\mu)$ and let $\{f_j : j\}$ be a net of μ-integrable functions from X into $\mathbf{K_s}$ converging to a function f uniformly on $\mathcal{R}_\mu$-compact subsets and such that $|f_j| \le |g|$ for every j. Then $f \in L(\mu)$ and $\lim_j \|f_j - f\|_\mu = 0$ and $\lim_j \int_X f_j(x)\mu(dx) = \int_X f(x)\mu(dx)$.

Remind also the equicontinuity theorem: let X be a topological vector space and let X' be its topologically dual; a subset H of X' is equicontinuous if and only if H is contained in a polar V^o of some neighborhood V of 0 in X (see (9.5.4) and Exer. 9.202 [NB85]).

Property (4) follows from Lemma 2.3, boundedness and continuity of χ_e and the fact that due to the Hahn-Banach theorem there is $x_z \in X$ with $z(x_z) = 1$ for $z \neq 0$ such that $z|_{(X\ominus L)} = 0$ and

$$\theta(z) = \int_X \chi_e(P_L(x))\mu(dx) = \int_L \chi_e(y)\mu_L(dy),$$

where $L = \mathbf{K}x_z$, also due to the Lebesgue theorem.

Indeed, for each $c > 0$ there exists a compact subset $S \subset X$ such that $\|X \setminus S\|_\mu < c$, each bounded subset $A \subset X^*$ is uniformly equicontinuous on S, that is, $\{\chi_e(z(x)) : z \in A\}$ is the uniformly equicontinuous family (by $x \in S$). On the other hand, $\chi_e(f(x))$ is uniformly equicontinuous on a bounded $A \subset C_b(X,\mathbf{K})$ by $x \in S$.

We call a functional θ finite-dimensionally concentrated, if there exists $L \subset X$, $dim_{\mathbf{K}} L < \aleph_0$, such that $\theta|_{(X\setminus L)} = \mu(X)$. For each $c > 0$ and $\delta > 0$ in view of Theorem 7.6[Roo78] recalled above there exists a finite-dimensional over $\mathbf{K}$ subspace L and compact $S \subset L^{\delta}$ such that $\|X \setminus S\|_{\mu} < c$. Let $\theta^L(z) := \theta(P_L z)$.

This definition is correct, since $L \subset X$, X has the isometrical embedding into X^* as the normed space associated with the fixed basis of X, such that functionals $z \in X$ separate points in X. If $z \in L$, then $|\theta(z) - \theta^L(z)| \le c \times b \times q$, where $b = \|X\|_{\mu}$, q is independent of c and b. Each characteristic functional $\theta^L(z)$ is uniformly continuous by $z \in L$ relative to the norm $\| * \|$ on L, since $|\theta^L(z) - \theta^L(y)| \le |\int_{S'\cap L}[\chi_e(z(x)) - \chi_e(y(x))]$ $\mu_L(dx)| + |\int_{L\setminus S'}[\chi_e(z(x)) - \chi_e(y(x))]\ \mu_L(dx)|$, where the second term does not exceed $2C'$ for $\|L \setminus S'\|_{\mu_L} < c'$ for a suitable compact subset $S' \subset X$ and $\chi_e(z(x))$ is an uniformly equicontinuous by $x \in S'$ family relative to $z \in B(L,0,1)$.

For a field $\mathbf{K}$ denote by $T_{\mathbf{K}}$ the group of all those roots of unity of $\mathbf{K}$ whose orders are not divisible by the characteristic p of the residue class field k of $\mathbf{K}$. A $\mathbf{K}$-valued character of a point-wise torsional group G is a continuous homomorphism $G \to T_{\mathbf{K}}$. The $\mathbf{K}$-valued characters form a group $\hat{G}_{\mathbf{K}}$.

Remind the basic theorem about the Fourier-Stieltjes transform (see also Theorem 9.20 [Roo78]): let G be a torsional group compatible with $\mathbf{K_s}$, then the Fourier-Stieltjes transform is an isomorphism of Banach algebras $M(G) \simeq BUC(\hat{G}_{\mathbf{K_s}})$.

A group G is supplied with a subgroup topology, if there exists a family $\mathcal{U}$ of its subgroups so that $\bigcap_{H\in\mathcal{U}} H = \{1\}$ and the co-sets of $H \in \mathcal{U}$ form a subbase for a zero-dimensional Hausdorff topology on G that renders G a topological group. Remind also that a topological group G with a subgroup topology is called torsional if every compact subset of G is contained in a compact subgroup of G.

Therefore,

$$\theta(z) = \lim_{n\to\infty} \theta_n(z) \tag{5}$$

for each finite-dimensional over $\mathbf{K}$ subspace L, where $\theta_n(z)$ is uniformly equicontinuous and finite-dimensionally concentrated on $L_n \subset X$, $z \in X$, $cl(\bigcup_n L_n) = X$, $L_n \subset L_{n+1}$ for every n, for each $c > 0$ there are n and $q > 0$ such that $|\theta(z) - \theta_j(z)| \le cbq$ for $z \in L_j$ and $j > n$, $q = const > 0$ is independent of j, c and b. Let $\{e_j : j \in \mathbf{N}\}$ be the standard orthonormal basis in X, $e_j = (0,\dots,0,1,0,\dots)$ with 1 in j-th place. Using Property 2.1.(iii) of μ, local constantness of χ_e, considering all $z = be_j$ and $b \in \mathbf{K}$, we get that $\theta(z)$ on X is non-trivial, whilst μ is a non-zero measure, since due to Lemma 2.3 μ is characterized uniquely by $\{\mu_{L_n} : n\}$. Indeed, for μ with values in $\mathbf{K_s}$ a measure μ_V on V, $dim_{\mathbf{K}} V < \aleph_0$, this follows from the basic theorem about the Fourier-Stieltjes transform, where

$$F(g)(z) := \lim_{r\to\infty} \int_{B(V,0,r)} \chi_e(z(x))g(x)m(dx),$$

$z \in V$, $g \in L(V,\mu_V,\mathbf{U_s})$, m is the Haar measure on V with values in $\mathbf{K_s}$. Therefore, the mapping $\mu \mapsto \theta_{\mu}$ is injective.

2.6. Theorem. *Let μ_1 and μ_2 be measures in $\mathsf{M}(X)$ on the same algebra E, where $Bco(X) \subset E \subset Bf(X)$ such that $\hat{\mu}_1(f) = \hat{\mu}_2(f)$ for each $f \in \Gamma$. Then $\mu_1 = \mu_2$, where*

$X = c_0(\alpha, K)$, $\alpha \leq \omega_0$, Γ *is a vector subspace in a space of continuous functions* $f : X \to \mathbf{K}$ *separating points in* X.

Proof. Let at first $\alpha < \omega_0$, then due to § 2.5 $\mu_1 = \mu_2$, since the family Γ generates E. Now let $\alpha = \omega_0$, $A = \{x \in X : (f_1(x), \ldots, f_n(x)) \in S\}$, ν_j be an image of a measure μ_j for a mapping $x \mapsto (f_1(x), \ldots, f_n(x))$, where $S \in E(\mathbf{K^n})$, $f_j \in X \hookrightarrow X^*$. Then $\hat{\nu}_1(y) = \hat{\mu}_1(y_1 f_1 + \cdots + y_n f_n) = \hat{\mu}_2(y_1 f_1 + \cdots + y_n f_n) = \hat{\nu}_2(y)$ for each $y = (y_1, \ldots, y_n) \in \mathbf{K^n}$, consequently, $\nu_1 = \nu_2$ on E. Further compositions of $f \in \Gamma$ with continuous functions $g : \mathbf{K} \to \mathbf{K_s}$ generate a family of $\mathbf{K_s}$-valued functions correspondingly separating points of X (see also Chapter 9 in [Roo78]).

2.7. Proposition. *Let* μ_l *and* μ *be measures in* $\mathsf{M}(X_l)$ *and* $\mathsf{M}(X)$ *respectively, where* $X_l = c_0(\alpha_l, K)$, $\alpha_l \leq \omega_0$, $X = \prod_1^n X_l$, $n \in \mathbf{N}$. *Then the condition* $\hat{\mu}(z_1, \ldots, z_n) = \prod_{l=1}^n \hat{\mu}_l(z_l)$ *for each* $(z_1, \ldots, z_n) \in X \hookrightarrow X^*$ *is equivalent to* $\mu = \prod_{l=1}^n \mu_l$.

Proof. Let $\mu = \prod_{l=1}^n \mu_l$, then $\hat{\mu}(z_1, \ldots, z_n) = \int_X \chi_e(\sum z_l(x_l)) \prod_{l=1}^n \mu_l(dx_l) = \prod_{l=1}^n \int_{X_l} \chi_e(z_l(x_l)) \mu_l(dx_l)$. The reverse statement follows from Theorem 2.6.

2.8. Proposition. *Let* X *be a Banach space over* $\mathbf{K}$*; suppose* μ, μ_1 *and* μ_2 *are probability measures on* X. *Then the following conditions are equivalent:* μ *is the convolution of two measures* μ_j, $\mu = \mu_1 * \mu_2$, *and* $\hat{\mu}(z) = \hat{\mu}_1(z)\hat{\mu}_2(z)$ *for each* $z \in X$.

Proof. Let $\mu = \mu_1 * \mu_2$. This means by the definition that μ is the image of the measure $\mu_1 \otimes \mu_2$ for the mapping $(x_1, x_2) \mapsto x_1 + x_2$, $x_j \in X$, consequently, $\hat{\mu}(z) = \int_{X \times X} \chi_e(z(x_1 + x_2))$ $(\mu_1 \otimes \mu_2)(d(x_1, x_2)) = \prod_{l=1}^2 \int_X \chi_e(z(x_l)) \mu_l(dx_l) = \hat{\mu}_1(z)\hat{\mu}_2(z)$. On the other hand, if $\hat{\mu}_1 \hat{\mu}_2 = \mu$, then $\hat{\mu} = (\mu_1 * \mu_2)^\wedge$ and due to the basic theorem about the Fourier-Stieltjes transform (see above) for measures with values in $\mathbf{K_s}$, we have $\mu = \mu_1 * \mu_2$.

2.9. Corollary. *Let* ν *be a probability measure on* $Bf(X)$ *and* $\mu * \nu = \mu$ *for each* μ *with values in the same field, then* $\nu = \delta_0$.

Proof. If $z_0 \in X \hookrightarrow X^*$ and $\hat{\mu}(z_0) \neq 0$, then from $\hat{\mu}(z_0)\hat{\nu}(z_0) = \hat{\mu}(z_0)$ it follows that $\hat{\nu}_0(z_0) = 1$. From Property 2.6(5) we get that there exists $m \in \mathbf{N}$ with $\hat{\mu}(z) \neq 0$ for each z with $\|z\| = p^{-m}$, since $\hat{\mu}(0) = 1$. Then $\hat{\nu}(z + z_0) = 1$, that is, $\hat{\nu}|_{(B(X, z_0, p^{-m}))} = 1$. Since μ are arbitrary we get $\hat{\nu}|_X = 1$, that is, $\nu = \delta_0$ due to § 2.5.

2.10. Corollary. *Let* X *and* Y *be Banach spaces over* $\mathbf{K}$, μ *and* ν *be probability measures on* X *and* Y *respectively, suppose* $T : X \to Y$ *is a continuous linear operator. A measure* ν *is an image of* μ *for* T *if and only if* $\hat{\nu} = \hat{\mu} \circ T^*$, *where* $T^* : Y^* \to X^*$ *is an adjoint operator.*

Proof. It follows immediately from § 2.5 and § 2.6.

2.11. Proposition. *For a completely regular space* X *with the zero small inductive dimension* $ind(X) = 0$ *the following statements are accomplished:*

(a). *if* (μ_β) *is a bounded net of measures in* $\mathsf{M}(X)$ *that weakly converges to a measure* μ *in* $\mathsf{M}(X)$, *then* $(\hat{\mu}_\beta(f))$ *converges to* $\hat{\mu}(f)$ *for each continuous* $f : X \to \mathbf{K}$*; if* X *is separable and metrizable then* $(\hat{\mu}_\beta)$ *converges to* $\hat{\mu}$ *uniformly on subsets that are uniformly equicontinuous in* $C(X, \mathbf{K})$*;*

(b). *if* M *is a bounded dense family in a ball of the space* $\mathsf{M}(X)$ *for measures in* $\mathsf{M}(X)$, *then a family* $(\hat{\mu} : \mu \in M)$ *is equicontinuous on a locally* $\mathbf{K}$*-convex space* $C(X, \mathbf{K})$ *in a topology of uniform convergence on compact subsets* $S \subset X$.

Proof. (a). Functions $\chi_e(f(x))$ are continuous and bounded on X, where $\hat{\mu}(f) = \int_X \chi_e(f(x)) \mu(dx)$. Then (a) follows from the definition of the weak convergence, since $span_{\mathbf{U_s}}\{\chi_e(f(x)) : f \in C(X, \mathbf{K}\}$ is dense in $C(X, \mathbf{U_s})$.

(b). For each $c > 0$ there exists a compact subset $S \subset X$ such that $\|\mu|_{(X \setminus S)}\| < c/4$ for

$\mathbf{K_s}$-valued measures. Therefore, for $\mu \in \mathsf{M}$ and $f \in C(X,\mathbf{K})$ with $|f(x)|_{\mathbf{K}} < c < 1$ for $x \in S$ we get $|\mu(X) - \hat{\mu}(f)| = |\int_X (1 - \chi_e(f(x))\mu(dx)| < c/2$ for $\mathbf{K_s}$-valued μ, since for $c < 1$ and $x \in S$ we have $\chi_e(f(x)) - \chi_e(-f(x)) = 0$.

2.12. Theorem. *Let X be a Banach space over $\mathbf{K}$, $\gamma : \Gamma \to \mathbf{C_s}$ be a continuous positive definite function, (μ_β) be a bounded weakly relatively compact net in the space $\mathsf{M}_t(X)$ of Radon norm-bounded measures and there exists $\lim_\beta \hat{\mu}_\beta(f) = \gamma(f)$ for each $f \in \Gamma$ and uniformly on compact subsets of the completion $\tilde{\Gamma}$, where $\Gamma \subset C(X,\mathbf{K})$ is a vector subspace separating points in X. Then (μ_β) weakly converges to $\mu \in \mathsf{M}_t(X)$ with $\hat{\mu}|_\Gamma = \gamma$.*

Proof. Is analogous to the proof of Theorem I.2.17 and follows from Theorem 2.6 and using the non-Archimedean Lebesgue convergence theorem recalled above.

2.13. Theorem. (a). *A bounded family of measures in $\mathsf{M}(\mathbf{K^n})$ is weakly relatively compact if and only if a family $(\hat{\mu} : \mu \in M)$ is equicontinuous on $\mathbf{K^n}$.*

(b). *If $(\mu_j : j \in \mathbf{N})$ is a bounded sequence of measures in $\mathsf{M}_t(\mathbf{K^n})$, $\gamma : \mathbf{K^n} \to \mathbf{U_s}$ is a continuous function, $\hat{\mu}_j(y) \to \gamma(y)$ for each $y \in \mathbf{K^n}$ uniformly on compact subsets in $\mathbf{K^n}$, then (μ_j) weakly converges to a measure μ with $\hat{\mu} = \gamma$.*

(c). *A bounded sequence of measures (μ_j) in $\mathsf{M}_t(\mathbf{K^n})$ weakly convereges to a measure μ in $\mathsf{M}_t(\mathbf{K^n})$ if and only if for each $y \in \mathbf{K^n}$ there exists $\lim_{j\to\infty} \hat{\mu}_j(y) = \hat{\mu}(y)$.*

(d). *If a bounded net (μ_β) in $\mathsf{M}_t(\mathbf{K^n})$ converges uniformly on each bounded subset in $\mathbf{K^n}$, then (μ_β) converges weakly to a measure μ in $\mathsf{M}_t(\mathbf{K^n})$, where $n \in \mathbf{N}$.*

Proof. (a). This follows from Proposition 2.11.

(b). Due to the non-Archimedean Fourier transform and the Lebesgue convergence theorem for $\mathbf{K_s}$-valued measures formulated above and from the condition $\lim_{R\to\infty} \sup_{|y|>R} |\gamma(y)|R^n = 0$ it follows, that for each $\varepsilon > 0$ there exists $R_0 > 0$ such that $\lim_m \sup_{j>m} \|\mu_j|_{\{x\in\mathbf{K^n}:|x|>R\}}\| \le 2\sup_{|y|>R} |\gamma(y)|R < \varepsilon$ for each $R > R_0$. In view of Theorem 2.12 (μ_j) converges weakly to μ with $\hat{\mu} = \gamma$. (c,d). These can be proved analogously to I.2.18.

2.14. Corollary. *If $(\hat{\mu}_\beta) \to 1$ uniformly on some neighborhood of 0 in $\mathbf{K^n}$ for a bounded net of measures μ_β in $\mathsf{M}_t(\mathbf{K^n})$, then (μ_β) converges weakly to δ_0.*

2.15. Definition. A family of probability measures $M \subset \mathsf{M}_t(X)$ for a Banach space X over $\mathbf{K}$ is called planely concentrated if for each $c > 0$ there exists a $\mathbf{K}$-linear subspace $S \subset X$ with $dim_{\mathbf{K}} S = n < \aleph_0$ such that $\inf(\|S^c\|_\mu : \mu \in M) > 1 - c$. The Banach space $\mathsf{M}_t(X)$ is supplied with the following norm $\|\mu\|$

2.16. Theorem. *Let X be a Banach space over $\mathbf{K}$ with a family $\Gamma \subset X^*$ separating points in $M \subset \mathsf{M}_t(X)$. Then M is weakly relatively compact if and only if a family $\{\mu_z : \mu \in M\}$ is weakly relatively compact for each $z \in \Gamma$ and M is planely concentrated, where μ_z is an image measure on $\mathbf{K}$ of a measure μ induced by z.*

Proof. It follows from the Alaoglu-Bourbaki theorem recalled above and Lemmas I.2.5 and I.2.21.

2.17. Theorem. *For X and Γ the same as in Theorem 2.16 a sequence $\{\mu_j : j \in \mathbf{N}\} \subset \mathsf{M}_t(X)$ is weakly convergent to $\mu \in \mathsf{M}_t(X)$ if and only if for each $z \in \Gamma$ there exists $\lim_{j\to\infty} \hat{\mu}_j(z) = \hat{\mu}(z)$ and a family $\{\mu_j\}$ is planely concentrated.*

Proof. It follows immediately from Theorems 2.12,13,16.

2.18. Proposition. *Let X be a weakly regular space with $ind(X) = 0$, $\Gamma \subset C(X,\mathbf{K})$ be a vector subspace separating points in X, $(\mu_n : n \in \mathbf{N}) \subset \mathsf{M}_t(X)$, $\mu \in \mathsf{M}_t(X)$, $\lim_{n\to\infty} \hat{\mu}_n(f) = \hat{\mu}(f)$ for each $f \in \Gamma$. Then (μ_n) is weakly convergent to μ relative to the weakest topology*

$\sigma(X,\Gamma)$ *in* X *relative to which all* $f \in \Gamma$ *are continuous.*

Proof. It follows from Theorem 2.13.

2.19. Let $(X,\mathsf{U}) = \prod_\lambda (X_\lambda, \mathsf{U}_\lambda)$ be a product of measurable completely regular Radon spaces $(X_\lambda, \mathsf{U}_\lambda) = (X_\lambda, \mathsf{U}_\lambda, \mathsf{K}_\lambda)$, where K_λ are compact classes approximating from below each measure μ_λ on $(X_\lambda, \mathsf{U}_\lambda)$, that is, for each $c > 0$ and elements A of an algebra U_λ there is $S \in \mathsf{K}_\lambda$, $S \subset A$ with $\|A \setminus S\|_{\mu_\lambda} < c$.

Theorem. *Each bounded quasi-measure* μ *with values in* $\mathbf{K_s}$ *on* (X,U) (*that is,* $\mu|_{\mathsf{U}_\lambda}$ *is a bounded measure for each* λ) *is extendible to a measure on an algebra* $Af(X,\mu) \supset \mathsf{U}$, *where an algebra* U *is generated by a family* $(\mathsf{U}_\lambda : \lambda \in \Lambda)$.

Proof. We have 2.1(i) by the condition and $\|X\|_\mu < \infty$, if 2.1(iii) is satisfied. It remains to prove 2.1(iii). For each sequence $(A_n) \subset \mathsf{U}$ with $\bigcap_n A_n = \emptyset$ and each $c > 0$ for each $j \in \mathbf{N}$ we choose $K_j \in \mathsf{K}$, where the compact class K is generated by (K_λ) (see also Proposition 1.1.8[DF91]), such that $K_j \subset A_j$ and $\|A_j \setminus K_j\|_\mu < c$. Since $\bigcap_{n=1}^\infty K_n \subset \bigcap_n A_n = \emptyset$, then there exists $l \in \mathbf{N}$ with $\bigcap_{n=1}^l K_n = \emptyset$, hence $A_l = A_l \setminus \bigcap_{n=1}^l K_n \subset \bigcup_{n=1}^l (A_n \setminus K_n)$, consequently, $\|A_l\|_\mu \le \max_{n=1,\dots,l}(\|A_n \setminus K_n\|_\mu) < c$.

Remind a theorem about uniqueness of extensions of measures (see also Theorem 7.8[Roo78]). Let μ be a measure on $\mathcal{R}$. Let $\mathcal{S}$ be a separating covering ring of X such that $\mathcal{S}$ is a sub-ring in $\mathcal{R}_\mu$ and let ν be a restriction of μ on $\mathcal{S}$. Then $\mathcal{S}_\nu = \mathcal{R}_\mu$ and $\bar{\nu} = \bar{\mu}$.

To finish the proof of this theorem it remains to use the theorem about uniqueness of an extension of a measure.

2.19.1. Note. More general theorem is given in § 2.37, since products are particular cases of projective limits (see also § § 2.36 and 2.38).

2.20. Definition. Let X be a Banach space over $\mathbf{K}$, then a mapping $f : X \to \mathbf{U_s}$ is called pseudo-continuous, if its restriction $f|_L$ is uniformly continuous for each $\mathbf{K}$-linear subspace $L \subset X$ with $dim_{\mathbf{K}} L < \aleph_0$. Let Γ be a family of mappings $f : Y \to \mathbf{K}$ of a set Y into a field $\mathbf{K}$. We denote by $\mathsf{C}(Y,\Gamma)$ an algebra of subsets of the form $C_{f_1,\dots,f_n;E} := \{x \in X : (f_1(x),\dots,f_n(x)) \in S\}$, where $S \in Bco(\mathbf{K^n})$, $f_j \in \Gamma$. We supply Y with a topology $\tau(Y)$ which is generated by a base $(C_{f_1,\dots,f_n;E} : f_j \in \Gamma$, E is open in $\mathbf{K^n})$.

2.21. Theorem. Non-Archimedean analog of the Bochner-Kolmogorov theorem. *Let* X *be a Banach space over* $\mathbf{K}$, X^a *be its algebraically dual* $\mathbf{K}$*-linear space* (*that is, of all linear mappings* $f : X \to \mathbf{K}$ *not necessarily continuous*). *A mapping* $\theta : X^a \to \mathbf{U_s}$ *is a characteristic functional of a probability measure* μ *with values in* $\mathbf{K_s}$ *and it is defined on* $C(X^a,X)$ *if and only if* θ *satisfies Conditions* 2.5(3,5) *for* $(X^a,\tau(X^a)$ *and is pseudo-continuous on* X^a, *where* $\mathbf{K_s} \cup \mathbf{C_s} \subset \mathbf{U_s}$.

Proof. (I). For $dim_{\mathbf{K}} X = card(\alpha) < \aleph_0$ a space X^a is isomorphic with $\mathbf{K}^\alpha$, hence the statement of this theorem for a measure μ with values in $\mathbf{K_s}$ follows from the basic theorem about the Fourier-Stieltjes transform reminded above and Theorems 2.6 and 2.13, since $\theta(0) = 1$ and $|\theta(z)| \le 1$ for each z.

(II). Now let $\alpha = \omega_0$. It remains to show that the conditions imposed on θ are sufficient, because their necessity follows from the modification of § 2.5 (since X has an algebraic embedding into X^a). The space X^a is isomorphic with $\mathbf{K}^\Lambda$ which is the space of all $\mathbf{K}$-valued functions defined on the Hamel basis Λ in X. Let J be a family of all non-void subsets in Λ. For each $A \in J$ there exists a functional $\theta_A : \mathbf{K}^A \to \mathbf{C}$ such that $\theta_A(t) = \theta(\sum_{y \in A} t(y) y)$ for $t \in \mathbf{K}^A$. From the conditions imposed on θ it follows that $\theta_A(0) = 1$, θ_A is uniformly continuous and bounded on $\mathbf{K}^A$, moreover, due to 2.5(5) for each $c > 0$ there are n and $q > 0$

such that for each $j > n$ and $z \in \mathbf{K}^A$ the following inequality is satisfied:

$$(i) \quad |\theta_A(z) - \theta_j(z)| \leq cbq,$$

moreover, $L_j \supset \mathbf{K}^A$, q is independent from j, c and b. From (I) it follows that on $Bf(\mathbf{K}^A)$ there exists a probability measure μ_A such that $\hat{\mu}_A = \theta_A$. The family of measures $\{\mu_A : A \in J\}$ is consistent and bounded, since $\mu_A = \mu_E \circ (P_E^A)^{-1}$, if $A \subset E$, where $P_E^A : \mathbf{K}^E \to \mathbf{K}^A$ are the natural projectors. Indeed, this is accomplished due to Conditions (i), 2.5(5) for X^a and due to the basic theorem about the Fourier-Stieltjes transform.

In view of Theorem 2.19 on a cylindrical algebra of the space $\mathbf{K}^\Lambda$ there exists the unique measure μ such that $\mu_A = \mu \circ (P^A)^{-1}$ for each $A \in J$, where $P^A : \mathbf{K}^\Lambda \to \mathbf{K}^A$ are the natural projectors. From $X^a = \mathbf{K}^\Lambda$ it follows that μ is defined on $C(X^a, X)$. For μ on $C(X^a, X)$ there exists its extension on $Af(X, \mu)$ such that $Af(X, \mu) \supset Bco(X)$ (see § 2.1).

2.22. For $f \in L(X, \mu, \mathbf{U_s})$ and $\mathbf{K_s}$-valued measure μ let

$$\int_X f(x)\mu_*(dx) = \lim_{n\to\infty} \int_X g_n(x)\mu_*(dx)$$

for norm-bounded sequence of cylindrical functions g_n from $L(X, \mu, \mathbf{U_s})$ converging to f uniformly on compact subsets of X, where $\mathbf{K_s} \subset \mathbf{U_s}$. Due to the Lebesgue converging theorem this limit exists and does not depend on a choice of $\{g_n : n\}$.

Lemma. *A sequence of weak distributions (μ_{L_n}) of probability Radon measures is generated by a $\mathbf{K_s}$-valued probability measure μ on $Bco(X)$ of a Banach space X over $\mathbf{K}$ if and only if there exists*

$$(i) \quad \lim_{|\xi|\to\infty} \int_X G_\xi(x)\mu_*(dx) = 1,$$

where $\int_X G_\xi(x)\mu_*(dx) := S_\xi(\{\mu_{L_n} : n\})$ and

$$S_\xi(\{\mu_{L_n}\}) := \lim_{n\to\infty} \int_{L_n} F_n(\gamma_{\xi,n})(x)\, \mu_{L_n}(dx), \quad \gamma_{\xi,n}(y) := \prod_{l=1}^{m(n)} \gamma_\xi(y_l),$$

F_n is a Fourier transformation by $(y_1, \dots, y_n)$, $y = (y_j : j \in \mathbf{N})$, $y_j \in \mathbf{K}$, $\gamma_\xi(y) := C(\xi)s^{-2\min(0, ord_p(y,\xi))}$, $C(\xi) \in \mathbf{K_s}$, $\gamma_\xi : \mathbf{K} \to \mathbf{K_s}$, $y, \xi \in \mathbf{K}$, $\nu_\xi(\mathbf{K}) = 1$, $\nu_\xi(dy) = \gamma_\xi(dy)w(dy)$, $w : Bco(\mathbf{K}) \to \mathbf{K_s}$ is the Haar measure; here $m(n) = dim_{\mathbf{K}} L_n < \aleph_0$, $cl(\bigcup_n L_n) = X = c_0(\omega_0, K)$.

Proof is quite analogous to that of § I.2.30 with the substitution of

$$\left| \int_X G_\xi(x)\mu_*(dx) - 1 \right| < c/2$$

for real-valued measures on $|\|G_\xi(x)\| - 1| < c/2$ for $\mathbf{K_s}$-valued measures.

2.23. Notes and definitions. Suppose X is a locally convex space over a locally compact field $\mathbf{K}$ with non-trivial non-Archimedean normalization and X^* is a topologically dual space. For a $\mathbf{K_s}$-valued measure μ on X a completion of a linear space of characteristic functions $\{ch_U : U \in Bco(X)\}$ in $L(X, \mu, \mathbf{U_s})$ is denoted by $B_\mu(X)$, where $\mathbf{K_s} \subset \mathbf{U_s}$. Then X is called a KS-space if on X^* there exists a topology τ such that the continuity of $f : X^* \to \mathbf{U_s}$ with $\|f\|_{C^0} < \infty$ is necessary and sufficient for f to be a characteristic functional of a tight measure of the finite norm, where $\mathbf{C_s} \subset \mathbf{U_s}$. Such topology is called the K-Sazonov type topology. The class of KS-spaces contains all separable locally convex spaces over $\mathbf{K}$. For example, $l^\infty(\alpha,\mathbf{K}) = c_0(\alpha,\mathbf{K})^*$. In particular we also write $c_0(\mathbf{K}) := c_0(\omega_0,\mathbf{K})$ and $l^\infty(\mathbf{K}) := l^\infty(\omega_0,\mathbf{K})$, where ω_0 is the first countable ordinal.

Let $n_{\mathbf{K}}(l^\infty,c_0)$ denotes the weakest topology on l^∞ for which all functionals $p_x(y) := \sup_n |x_n y_n|$ are continuous, where $x = \sum_n x_n e_n \in c_0$ and $y = \sum_n y_n e_n^* \in l^\infty$, e_n is the standard base in c_0. Such topology $n_{\mathbf{K}}(l^\infty,c_0)$ is called the normal topology. The induced topology on c_0 is denoted by $n_{\mathbf{K}}(c_0,c_0)$.

2.23.1. Proposition. (i). *The topology $n_{\mathbf{K}}(l^\infty,c_0)$ is the $\mathbf{K}$-locally convex Hausdorff topology on $l^\infty(\mathbf{K})$ and the family of sets $\mathcal{U} := \{U_x : x \in c_0\}$, where $U_x := \{z \in l^\infty : p_x(z) < 1\}$, is the base of neighborhoods of zero for it.*

(ii). *The topology $n_{\mathbf{K}}(l^\infty,c_0)$ is strictly weaker than the norm topology in l^∞.*

(iii). *The space c_0 is dense in l^∞ in the topology $n_{\mathbf{K}}(l^\infty,c_0)$.*

Proof. (i). If G is a T_0-group, then G is a Hausdorff group (see Chapter 1 in [HR79]). Since l^∞ has the additive group structure and $n_{\mathbf{K}}(l^\infty,c_0)$ is the T_0-topology, hence l^∞ is Hausdorff in this topology. By the definition $\mathcal{U}$ forms the base of neighborhoods of zero.

(ii). To prove it consider the standard orthonormal base $\{e_n : n\}$ in c_0, then $\lim_{n\to\infty} p_x(e_n) = |x_n|_{\mathbf{K}} = 0$ for each $x = (x_n : n) \in c_0$, where $x_n \in \mathbf{K}$ for each $n \in \mathbf{N}$.

(iii). Put $a_n(x) := (x_1,\dots,x_n,0,0,\dots)$ for each $x \in l^\infty$. Then

$\lim_{n\to\infty} p_z(a_n(x)-x) = \lim_{n\to\infty} \sup_{n\in\mathbf{N}} |x_{n+1}y_{n+1}|_{\mathbf{K}} = 0$

for each $y \in c_0$.

2.24. Theorem. *Let $f : l^\infty(\mathbf{K}) \to \mathbf{U_s}$ be a functional such that*

(i) *$f(0) = 1$ and $\|f\|_{C^0} \le 1$,*

(ii) *f is continuous in the normal topology $n_{\mathbf{K}}(l^\infty,c_0)$, then f is the characteristic functional of a probability measure on $c_0(\mathbf{K})$.*

Proof. If ν is the Haar measure on $\mathbf{K^n}$, then on $Bco(\mathbf{K^n})$ it takes values in $\mathbf{Q}$, when it is chosen with $\nu(B(\mathbf{K^n},0,1)) = 1$ and hence $\nu(B(\mathbf{K^n},x,p^n)) = p^n$ for each $x \in \mathbf{K^n}$ and each $n \in \mathbf{Z}$. If $\mathbf{K}$ is a locally compact non-Archimedean field, R is its maximal compact sub-ring $B(\mathbf{K},0,1)$ and P is the maximal ideal of R, then the order of a nontrivial character χ of $\mathbf{K}$ is the largest integer $n \in \mathbf{Z}$ such that $\chi|_{P^{-n}} = 1$, it is denoted by $ord(\chi) := n$. In view of Proposition 12 [Wei73] if χ is a non-trivial character of $\mathbf{K}$ of order n, then for each $m \in \mathbf{Z}$, $\chi(xt) = 1$ for each $t \in P^m$ if and only if $x \in P^{-n-m}$. Therefore, for a characteristic function Ch_B of a ball B we have

$$Ch_{B(\mathbf{K^n},0,|\pi|^m)}(x_1,\dots,x_n) = [\nu(B(\mathbf{K^n},0,|\pi|^{-m}))]^{-1}$$

$$\int_{\mathbf{K^n}} Ch_{B(\mathbf{K^n},0,|\pi|^{-m})}(y_1,\dots,y_n)\chi\left(\sum_{l=1}^n x_l y_l\right)\nu(dy),$$

since $B(\mathbf{K^n},0,|\pi|^{-m}) = (B(\mathbf{K},0,|\pi|^{-m}))^n$, where $P = \pi R$, $|\pi|$ is the generator of the normalization group $\Gamma_{\mathbf{K}}$. Therefore,

$$P\{|V_1|_{\mathbf{K}} < |\pi|^m,\dots,|V_n|_{\mathbf{K}} < |\pi|^m\} = \int_{\mathbf{K^n}} Ch_{B(\mathbf{K^n},0,|\pi|^m)}(V_1(\omega),\dots,V_n(\omega))P(d\omega)$$

$$= \int_\Omega \left\{[\nu(B(\mathbf{K^n},0,|\pi|^{-m}))]^{-1}\int_{\mathbf{K^n}} Ch_{B(\mathbf{K^n},0,|\pi|^{-m})}(y)\chi\left(\sum_{l=1}^n y_l V_l(\omega)\right)\nu(dy)\right\}P(d\omega),$$

hence

$$(i)\ \ P\{|V_1|_{\mathbf{K}} < |\pi|^m, \dots, |V_n|_{\mathbf{K}} < |\pi|^m\}$$

$$= [\nu(B(\mathbf{K^n}, 0, |\pi|^{-m}))]^{-1} \int_{\mathbf{K^n}} f_W(y) Ch_{B(\mathbf{K^n},0,|\pi|^{-m})}(y)\nu(dy)$$

for measurable maps $V_j : (\Omega, \mathsf{B}, P) \to (\mathbf{K}, Bco(\mathbf{K}))$, where (Ω, B, P) is a probability space for a probability measure P with values in $\mathbf{K_s}$ on an algebra B of subsets of a set Ω, f_W is a characteristic function of $W = (V_1, \dots, V_n)$, $f_W(y) := \int_\Omega \chi_{W(\omega)}(y) P(d\omega)$. To continue the proof we need the following statements.

2.25. Lemma *Let $f : c_0(\mathbf{K}) \to \mathbf{C_s}$ be a function satisfying the following two conditions:*

(i) $|f(x)| \le 1$ *for each* $x \in c_0(\mathbf{K})$,

(ii) f *is continuous at zero in the topology* $n_{\mathbf{K}}(c_0, c_0)$,

then for each $\varepsilon > 0$ *there exists* $\lambda(\varepsilon) \in c_0(\mathbf{K})$ *such that* $|1 - f(x)| < p_{\lambda(\varepsilon)}(x) + \varepsilon$ *for each* $x \in c_0(\mathbf{K})$.

Proof. In view of continuity for each $\varepsilon > 0$ there exists $y(\varepsilon) \in c_0$ such that $|1 - f(x)| < \varepsilon$ if $p_{y(\varepsilon)} < 1$. Put $\lambda(\varepsilon) = \pi_{\mathbf{K}}^{-1} y(\varepsilon)$, where $\pi_{\mathbf{K}} \in \mathbf{K}$ is such that $|\pi_{\mathbf{K}}| = p^{-1}$. If $x \in c_0$ is such that $p_{\lambda(\varepsilon)}(x) < p^{-1}$, then $|1 - f(x)| < \varepsilon \le \varepsilon + p_{\lambda(\varepsilon)}(x)$. If $p_{\lambda(\varepsilon)}(x) \ge p$, then $|1 - f(x)| \le 2 \le p < p_{\lambda(\varepsilon)}(x) + \varepsilon$.

2.26. Lemma. *Let $\{V_n : n \in \mathbf{N}\}$ be a sequence of $\mathbf{K}$-valued random variables for P with values in $\mathbf{K_s}$. If for each $\beta > 0$ and $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbf{N}$ such that*

$$(i)\ \ \|P|_{\{\sup_{n \ge N_\varepsilon} |V_n|_{\mathbf{K}} \le \beta\}}\| \ge 1 - \varepsilon(1 + \beta^{-1}),$$

then $\lim_n V_n = 0$ *P-almost everywhere on* Ω.

Proof. Consider a marked $\beta > 0$ and for each $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbf{N}$ such that Inequality (i) is satisfied. Take a sequence $\{\varepsilon_n : n \in \mathbf{N}\}$ such that $0 < \varepsilon_{n+1} < \varepsilon_n$ for each n and $\lim_{n\to\infty} \varepsilon_n = 0$, then there exists a sequence $\{k_n := N_{\varepsilon_n} : n\}$ such that $k_{n+1} \ge k_n$ for each $n \in \mathbf{N}$. Put $a_n(\omega) := \sup_{m \ge k_n} |V_n(\omega)|_{\mathbf{K}}$. Then $a_n \ge 0$ and $a_{n+1} \le a_n$ for each $n \in \mathbf{N}$. Hence there exists $\lim_{n\to\infty} a_n(\omega) = \inf_n \sup_{m \ge k_n} |V_n(\omega)|_{\mathbf{K}} =: X(\omega)$. Since $\Gamma_{\mathbf{K}}$ is discrete in $(0, \infty)$ for the local field $\mathbf{K}$, then $X(\omega)$ is the discrete real random variable on (Ω, B, P). Consider $b_n(\omega) := \sup_{m \ge n} |V_m(\omega)|_{\mathbf{K}}$, consequently, $b_n \ge 0$ and $b_{n+1} \le b_n$ for each $n \in \mathbf{N}$. Thus there exists $\lim_{n\to\infty} b_n$ and $\{a_n : n\}$ is the subsequence of $\{b_n : n\}$, hence $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = X$. From the definition of a_n we have $\|P|_{a_n \le \beta}\| \ge 1 - \varepsilon_n(1 + \beta^{-1})$. Since P is the probability measure, then $\|P|_{(\lim_{n\to\infty} a_n) \le \beta}\| = 1$ and $\|P|_{(\lim_{n\to\infty} b_n) \le \beta}\| = 1$ for each $\beta > 0$. Therefore, $\|P|_{(\lim_{n\to\infty} b_n = 0)}\| = 1$ and inevitably $\lim_{n\to\infty} b_n = 0$ P-almost everywhere.

2.27. Proposition. *Let $f : c_0(\mathbf{K}) \to \mathbf{U_s}$ be a function such that*

(i) $f(0) = 1$ *and* $|f(x)| \le 1$ *for each* $x \in c_0$,

(ii) $f(x)$ *is continuous in the normal topology* $n_{\mathbf{K}}(c_0, c_0)$. *Then there exists a probability measure μ on $c_0(\mathbf{K})$ such that $f(x) = \hat{\mu}(x)$ for each $x \in c_0$.*

Proof. Consider functions $f_n(x_1, \dots, x_n) := f(x_1 e_1 + \dots + x_n e_n)$, where $x = \sum_j x_j e_j \in c_0$. From Condition (ii) and Proposition 2.23.1.(ii) it follows, that $f(x)$ is continuous in the norm topology.

The field $\mathbf{K}$ is spherically complete and X has a basis orthonormal in the non-Archimedean sense. So there are canonical embeddings of $\mathbf{K^n}$ into X and the corresponding projections from X onto such finite dimensional over $\mathbf{K}$ subspaces associated with the basis.

From Chapters 7,9 [Roo78] it follows, that there exists a consistent family of tight measures μ_n on $\mathbf{K^n}$ such that $\hat{\mu}_n(x) = f_n(x)$ for each $x \in \mathbf{K^n}$. In view of Theorem 2.19 there exists a probability space (Ω, B, P) with a $\mathbf{K_s}$-valued measure P and a sequence of random variables $\{V_n\}$ such that $\mu_n(A) = P\{\omega \in \Omega : (V_1(\omega), \dots, V_n(\omega)) \in A\}$ for each clopen subset A in $\mathbf{K^n}$, consequently, $\lim_n V_n = 0$ P-almost everywhere in Ω. In view of the preceding lemmas we have the following inequality:

$$|1 - \|P|_{(|V_n|<\beta,\dots,|V_{n+m}|<\beta)}\| \le \|p_{\lambda(\varepsilon)}(y_1 e_n + \dots + y_m e_{n+m}\|_{L(B(\mathbf{K^n},0,\beta^{-1}),\nu,\mathbf{K_s})}.$$

Since $\lim_k p_{\lambda(\varepsilon)}(e_k) = 0$, then there exists $N \in \mathbf{N}$ such that $\sup_{k\ge N} p_{\lambda(\varepsilon)}(e_k) \le \varepsilon$, consequently, $\|P|_{\{|V_N|<\beta,\dots,|V_{N+m}||<\beta\}}\| \ge 1 - \varepsilon(1+\beta^{-1})$. Due to Lemma 2.34 $\|P|_{\{\lim_n V_n = 0\}}\| = 1$. Define a measurable mapping W from Ω into c_0 by the following formula: $W(\omega) := \sum_n V_n(\omega) e_n$ for each $\omega \in \Omega$, then we also define a measure $\mu(A) := P\{W^{-1}(B)\}$ for each $A \in Bco(X)$, hence μ is a probability measure on c_0. In view of the Lebesgue convergence theorem there exists $\hat{\mu}(x) = \lim_n \hat{\mu}_n(x_1 e_1 + \dots + x_n e_n) = f(x)$ for each $x \in c_0$.

Continuation of the proof of Theorem 2.24. Let $f : l^\infty(\mathbf{K}) \to \mathbf{C_s}$ satisfies assumption of Theorem 2.24, then by Proposition 2.27 there exists a probability measure μ on $c_0(\mathbf{K})$ such that $f(x) = \hat{\mu}(x)$ for each $x \in c_0(\mathbf{K})$.

2.28. Theorem. *Let μ be a probability measure on $c_0(\mathbf{K})$, then $\hat{\mu}$ is continuous in the normal topology $n_{\mathbf{K}}(l^\infty, c_0)$ on l^∞.*

Proof. Let μ be a probability measure on c_0. Then in view of Lemma 2.3 for each $0 < \varepsilon < 1$ there exists a sequence $\{a_n : n\} \in c_0$ and a compact subset $K := K_{\{a_n:n\}} := \{x \in c_0 : |x_n| \le |a_n| \quad \forall n \in \mathbf{N}\}$ for which $\|\mu|_{c_0 \setminus K}\| < \varepsilon$. Therefore, $1 - \hat{\mu}(z) = (\int_{c_0\setminus K} + \int_K)(1 - \chi_e(z(x))\mu(dx))$, where $\sup_{x\in c_0\setminus K} |1 - \chi_e(z(x))| N_\mu(x) < \varepsilon$. Since χ_e is continuous and locally constant, $\chi_e(0) = 1$, K is compact, hence χ_e is uniformly continuous on K and there exists a constant $\delta > 0$ such that $\sup_{y\in K} |1 - \chi_e(z(y))| \le \delta \sup_{y\in K} |z(y)|$. Therefore, $|1 - \hat{\mu}(z)| \le |\int_K (1 - \chi_e(z(x)))\mu(dx)| \le \varepsilon + \delta \sup_{y\in K} |z(y)| \le \varepsilon + \sup_{n\in\mathbf{N}} |z_n a_n| \le \varepsilon + p_a(z)$ for each $z \in l^\infty$.

2.29. Corollary. *The normal topology $n_{\mathbf{K}}(l^\infty, c_0)$ is the K-Sazonov type topology on $l^\infty(\mathbf{K})$.*

2.29.1. Remark. Since each Banach space X of separable type over a locally compact non-Archimedean field is isomorphic with c_0 (see the theorems at the beginning of Chapter I above or Chapters 4 and 5 in [Roo78]), then from Corollary 2.29 it follows, that $n_{\mathbf{K}}(X^*, X)$ is the Sazonov type topology on X^*.

2.30. Theorem. Non-Archimedean analog of the Minlos-Sazonov theorem. *For a separable Banach space X over $\mathbf{K}$ the following two conditions are equivalent:*

$$(I)\ \theta : X \to \mathbf{U_s} \textit{ satisfies Conditions } 2.5(3,5) \textit{ and}$$

for each $c > 0$ there exists a compact operator $S_c : X \to X$ such that $|\theta(y) - \theta(x)| < c$ for $|\tilde{z}(S_c z)| < 1$;

$$(II)\ \theta \textit{ is a characteristic functional of a probability Radon measure } \mu$$

on E, where $\tilde{z}$ is an element $z \in X \hookrightarrow X^*$ considered as an element of X^* under the natural embedding associated with the standard base of $c_0(\omega_0, \mathbf{K})$, $z = x - y$, x and y are arbitrary elements of X.

Proof. $(II \to I)$. For θ generated by a $\mathbf{K_s}$-valued measure for each $r > 0$ we have $|\theta(0) - \theta(x)| = |\int_X (1 - \chi_e(x(u)))\mu(du)| \le \|(1 - \chi_e(x(u)))|_{B(X,0,r)}\|_\mu + 2\|\mu|_{(X\setminus B(X,0,r))}\|$. In view of the Radon property of the space X and Lemma I.2.5 for each $b > 0$ and $\delta > 0$ there are a finite-dimensional over $\mathbf{K}$ subspace L in X and a compact subset $W \subset X$ such that $W \subset L^\delta$, $\|\mu|_{(X\setminus W)}\| < b$, hence $\|\mu|_{(X\setminus L^\delta)}\| < b$.

We consider the expression $J(j,l)$ (see § I.2.35). and the compact operator $S : X \to X$ with $\tilde{e}_j(Se_l) = \xi_{j,l}t$. Then $|\theta(0) - \theta(z)| < c/2 + |\tilde{z}(Sz)| < c$ for the $\mathbf{K_s}$-valued measure, if $|\tilde{z}(Sz)| < |t|c/2$, where $\mathbf{Q_s} \subset \mathbf{K_s} \subset \mathbf{U_s}$. We choose r such that $\|\mu|_{(X\setminus B(X,0,r))}\| < c/2$ with S corresponding to $(r_j : j)$, where $r_1 = r$, $L_1 = L$, then we take $t \in \mathbf{K}$ with $|t|c = 2$.

$(I \to II)$. Without restriction of generality we may take $\theta(0) = 1$ after renormalization of non-trivial θ. In view of Theorem 2.24 as in § 2.5 we construct using $\theta(z)$ a consistent family of finite-dimensional distributions $\{\mu_{L_n} : n\}$ all with values in $\mathbf{K_s}$. Let m_{L_n} be the $\mathbf{K_s}$-valued Haar measure on L_n which is considered as $\mathbf{Q_p^a}$ with $a = dim_{\mathbf{K}}L_n dim_{\mathbf{Q_p}}\mathbf{K}$, $m(B(L_n,0,1)) = 1$. If S_c is a compact operator such that $|\theta(y) - \theta(x)| < c$ for $|\tilde{z}(S_c z)| < 1$, $z = x - y$, then $|1 - \theta(x)| < \max(C, 2|\tilde{x}(S_c x)|)$ and $\|\gamma_{\xi,n}(z)(1 - \theta(z))\|_{m_{L_n}} \le$
$\max(\|\gamma_{\xi,n}(z)\|_{m_{L_n}}C, 2|(\gamma_{\xi,n}(z))\tilde{z}(S_c z)|_{m_{L_n}}) \le \max(C, b\|S_c\|/|\xi|^2)$,
where $b := p \times \sup_{|\xi|>r}(|\xi|^2\|\gamma_{\xi,n}(z)z^2\|_{m_{L_n}}) < \infty$ for the $\mathbf{K_s}$-valued measures.

Remind the substitution theorem for integrals (see also [Kob77, Roo78, Sch84, Wei73]). Let U an V be compact open subsets in $\mathbf{K}$ and let $\sigma : U \to V$ be a C^1-homeomorphism, $\sigma'(x) \neq 0$ for all $x \in U$. Let $f : V \to \mathbf{K_s}$ be a continuous function, $\lambda : Bco(\mathbf{K}) \to \mathbf{K_s}$ be a Haar measure, then $\int_V f(y)\lambda(dy) = \int_U f(\sigma(x)) mod_{\mathbf{K}}(\sigma'(x))\lambda(dx)$.

Due to the formula of changing variables in integrals the following equality is valid:

$$|1 - \|G_\xi(x)\|_{\mu_*}| \le \max(C, b\|S_c\|/|\xi|^2)$$

for the $\mathbf{K_s}$-valued measures. Then taking the limit with $|\xi| \to \infty$ and then with $c \to +0$ with the help of Lemma 2.22 we get the statement $(I \to II)$.

2.31. Definition. Let on a completely regular space X with $ind(X) = 0$ two non-zero $\mathbf{K_s}$-valued measures μ and ν are given. Then ν is called absolutely continuous relative to μ if there exists f such that $\nu(A) = \int_A f(x)\mu(dx)$ for each $A \in Bco(X)$, where $f \in L(X,\mu,\mathbf{K_s})$ and it is denoted $\nu \ll \mu$. Measures ν and μ are singular to each other if there is $F \in E$ with $\|X \setminus F\|_\mu = 0$ and $\|F\|_\nu = 0$ and it is denoted $\nu \perp \mu$. If $\nu \ll \mu$ and $\mu \ll \nu$ then they are called equivalent, $\nu \sim \mu$.

2.32. Definition and note. For $\mu : E(X) \to \mathbf{K_s}$ a sequence $(\phi_n(x) : n) \subset L(\mu)$ is called a martingale if for each $\psi \in L(\mu|\mathsf{U}_n)$:

$$(i)\ \int_X \phi_{n+1}(x)\psi(x)\mu(dx) = \int_X \phi_n(x)\psi(x)\mu(dx)$$

such that $(\phi_n : n)$ is uniformly converging on $Af(X,\mu)$-compact subsets in X, where U_n is the minimal algebra such that $(\phi_j : j = 1,\dots,n) \subset L(\mu|\mathsf{U}_n)$, $\mu|\mathsf{U}_n$ is a restriction of μ on $\mathsf{U}_n \subset E(X)$, X is the Banach space over $\mathbf{K}$.

Recall Lemma 7.10 [Roo78]. Let μ be a measure on $\mathcal{R}$. For $b > 0$ put $X_b := \{x : N_\mu(x) \ge b\}$. Then the restrictions of the $\mathcal{R}$- and the $\mathcal{R}_\mu$-topologies on X_b coincide. A function $f : X \to \mathbf{K}$ is $\mathcal{R}_\mu$-continuous if and only if for every $b > 0$ the restriction of f to X_b is $\mathcal{R}$-continuous.

Remind as well Theorem 7.12 [Roo78]. Let μ be a measure on $\mathcal{R}$. A function $f : X \to \mathbf{K}$ is μ-integrable if and only if it satisfies the following two properties.

(Ri). f is $\mathcal{R}_\mu$-continuous.

(Rii). For each $b > 0$ the set $\{x : |f(x)|N_\mu(x) \geq b\}$ is $\mathcal{R}_\mu$-compact, hence contained in some $\{x : N_\mu(x) \geq \delta\}$ with $\delta > 0$.

In view of these two statements formulated just above for $\|X\|_\mu < \infty$ the $Af(X,\mu)$-topology on compact subspaces $X_c := [x \in X : N_\mu(x) \geq c]$ coincides with the initial topology, if μ is defined on E such that $Bco(X) \subset E \subset Af(X,\mu)$, where $c > 0$.

2.33. Theorem. *If there is a martingale* $(\phi_n : n)$ *for* μ *with values in* $\mathbf{K_s}$ *and* $\sup_n \|\phi_n\|_{N_\mu} < \infty$, *then there exists* $\lim_{n\to\infty} \phi_n(x) =: \phi(x) \in L(\mu)$.

Proof. Let $\psi(x)$ be a characteristic function of a clopen subset in X, then for each ϕ_n there exists a sequence of simple functions $(\phi_n^j : j \in \mathbf{N})$ such that $\lim_{j\to\infty} \|\phi_n - \phi_n^j\|_{N_\mu} = 0$. From $\|\phi_n - \phi_n^{j(n)}\|_{N_\mu} < c$ and 2.32.(i) it follows that $|\int_X (\phi_{n+1}^{j(n+1)}(x) - \phi_n^{j(n)})\psi(x)\mu(dx)| < c\|\psi\|_{N_\mu}$ for each $\psi \in L(\mu)$, consequently, $\|\phi_{n+1}^{j(n+1)} - \phi_n^{j(n)}\|_{N_{mu}} < c$ and there exists $\lim_{n\to\infty} \phi_n^{j(n)} = \lim_{n\to\infty} \phi_n = \phi \in L(\mu)$ due to the Lebesgue theorem, if $(c = c(n) = s^{-n} : n \in \mathbf{N})$, where for each ϕ_n is chosen $j(n) \in \mathbf{N}$, since $(\phi_n^{j(n)} : n)$ is a Cauchy sequence in the Banach space $L(\mu)$ due to the ultra-metric inequality.

2.34. Proposition. *Let* $(X, \mathcal{R}, \mu)$ *be a measure space. Then there exists a quotient mapping* $\pi : X \to Y$ *on a Hausdorff zero-dimensional space* $(Y, \tau_{\mathcal{G}})$ *and* $\pi(\mu) := \nu$ *is a measure on* Y *such that* $\mathcal{G} = \pi(\mathcal{R})$, *where* $(Y, \mathcal{G}, \nu)$ *is a measure space.*

Proof. Suppose that $(Y, \tau_{\mathcal{G}})$ is a T_0-space, where $\mathcal{G}$ is a covering ring of Y. For each two distinct points x and y in Y there exists a clopen subset U in Y such that either $x \in U$ and $y \in Y \setminus U$ or $y \in U$ and $x \in Y \setminus U$, since the base of topology $\tau_{\mathcal{G}}$ in Y consists of clopen subsets. On the other hand, $Y \setminus U$ is also clopen, since U is clopen. Therefore, Y is the Hausdorff space. Clearly this implies that Y is the Tychonoff space (see also § 6.2 [Eng86], but it is necessary to note that we consider the definition of the zero-dimensional space more general without T_1-condition in § 2.1.2).

Now we construct a T_1-space Y, that is a quotient space of X. For this consider the relation in X:
$x\kappa y$ if and only if for each $S \in \mathcal{R}$ with $x \in S$ there is the inclusion $\{x, y\} \subset S$. Evidently, $x\kappa x$, that is, κ is reflexive. The relation $x\kappa y$ means, that $y \in V_x := \bigcap_{x\in S\in\mathcal{R}} S$, where V_x is closed in X, then from $y \in S$ it follows, that $x \in S$, since otherwise $y \notin V_x$, because $\mathcal{R}$ is a covering ring. Therefore, $V_x = V_y$ and $y\kappa x$, hence κ is symmetric. Let $x\kappa y$ and $y\kappa z$, then $V_x = V_y = V_z$, consequently, $x\kappa z$ and κ is transitive. Therefore, κ is the equivalence relation. Let $\pi : X \to Y := X/\kappa$ be the quotient mapping and Y be supplied with the zero-dimensional topology generated by the covering ring $\mathcal{G}$ such that $\pi^{-1}(\mathcal{G}) = \mathcal{R}$, since each $A \in \mathcal{R}$ is clopen, then for each $x \in A \in \mathcal{R}$ we have $V_x \subset A$. Then $\pi^{-1}([y]) = V_y$ for each $y \in X$ and $[y] := \pi(y)$. Hence each point $[y] \in Y$ is closed, hence Y is the T_1-space. The topology in Y is generated by the covering ring $\mathcal{G}$, consequently, Y is the Hausdorff space (see above), since from the T_1 separation property it follows the T_0 separation property.

If $\mathcal{S}$ is a shrinking family with the void intersection in Y such that $\mathcal{S} \subset \mathcal{G}$, then $\pi^{-1}(\mathcal{S})$ is also a shrinking family with the void intersection in X such that $\pi^{-1}(\mathcal{S}) \subset \mathcal{R}$, hence from $\lim_{A\in\pi^{-1}(\mathcal{S})} \mu(A) = 0$ it follows $\lim_{A\in\mathcal{S}} \nu(A) = 0$. Therefore, Condition (iii) from § 2.1 is satisfied. Evidently, $\|\nu\| = \|\mu\|$ and ν is additive on $\mathcal{G}$, hence ν is the measure.

2.35. Note. In view of Proposition 2.34 we consider henceforth Hausdorff zero-dimensional measurable $(X, \mathcal{R})$ spaces if another is not specified.

In the classical case the principal role in stochastic analysis plays the Kolmogorov theorem, that gives the possibility to construct a stochastic process on the basis of a system of finite dimensional real-valued probability distributions (see § III.4 [Kol56]). The following theorems resolve this problem for $\mathbf{K_s}$-valued measures in cases of a product of measure spaces, a consistent family of measure spaces and in cases of bounded cylindrical distributions.

Consider now a family of probability measure spaces $\{(X_j, \mathcal{R}_j, \mu_j) : j \in \Lambda\}$, where Λ is a set. Suppose that each covering ring $\mathcal{R}_j$ is complete relative to a measure μ_j, that is, $\mathcal{R}_j = \mathcal{R}_{\mu_j}$, where $\mathcal{R}_{\mu_j}$ denotes a completion of $\mathcal{R}_j$ relative to μ_j. Let $X := \prod_{j\in\Lambda} X_j$ be the product of topological spaces supplied with the product, that is Tychonoff, topology τ_X, where each X_j is considered in its $\tau_{\mathcal{R}_j}$-topology. There is the natural continuous projection $\pi_j : X \to X_j$ for each $j \in \Lambda$. Let $\mathcal{R}$ be the ring of the form $\bigcup_{j_1,\dots,j_n\in\Lambda, n\in\mathbf{N}} \bigcap_{l=1}^{n} \pi_{j_l}^{-1}(\mathcal{R}_{j_l})$.

2.36. Note. A set Λ is called directed if there exists a relation $\le$ on it satisfying the following conditions:

$(D1)$ If $j \le k$ and $k \le m$, then $j \le m$;

$(D2)$ For every $j \in \Lambda$, $j \le j$;

$(D3)$ For each j and k in Λ there exists $m \in \Lambda$ such that $j \le m$ and $k \le m$. A subset Υ of Λ directed by $\le$ is called co-final in Λ if for each $j \in \Lambda$ there exists $m \in \Upsilon$ such that $j \le m$. Suppose that Λ is a directed set and $\{(X_j, \mathcal{R}_j, \mu_j) : j \in \Lambda\}$ is a family of probability measure spaces, where $\mathcal{R}_j$ is the covering ring (not necessarily separating). Supply each X_j with a topology τ_j such that its base is a ring $\mathcal{R}_j$. Let this family be consistent in the following sense:

(1) there exists a mapping $\pi_j^k : X_k \to X_j$ for each $k \ge j$ in Λ such that $(\pi_j^k)^{-1}(\mathcal{R}_j) \subset \mathcal{R}_k$, $\pi_j^j(x) = x$ for each $x \in X_j$ and each $j \in \Lambda$, $\pi_k^m \circ \pi_l^k = \pi_l^m$ for each $m \ge k \ge l$ in Λ;

(2) $\pi_l^k(\mu_k) = (\mu_l)$ for each $k \ge l$ in Λ. Such family of measure spaces is called consistent.

2.37. Theorem. *Let $\{(X_j, \mathcal{R}_j, \mu_j) : j \in \Lambda\}$ be a consistent family as in* § 2.36. *Then there exists a probability measure space $(X, \mathcal{R}_\mu, \mu)$ and a mapping $\pi_j : X \to X_j$ for each $j \in \Lambda$ such that $(\pi_j)^{-1}(\mathcal{R}_j) \subset \mathcal{R}$ and $\pi_j(\mu) = \mu_j$ for each $j \in \Lambda$.*

Proof. We have $(\pi_j^k)^{-1}(\mathcal{R}_j) \subset \mathcal{R}_k$ for each $k \ge j$ in Λ, then $(\pi_j^k)^{-1}(\tau_j) \subset \tau_k$ for each $k \ge j$ in Λ, since each open subset in (X_j, τ_j) is a union of some subfamily $\mathcal{G}$ in $\mathcal{R}_j$ and $(\pi_j^k)^{-1}(\bigcup \mathcal{G}) = \bigcup_{A\in\mathcal{G}} (\pi_j^k)^{-1}(A)$. Therefore, each π_j^k is continuous and there exists the inverse system $\mathsf{S} := \{X_k, \pi_j^k, \Lambda\}$ of the spaces X_k. Its limit $\lim \mathsf{S} =: X$ is the topological space with a topology τ_X and continuous mappings $\pi_j : X \to X_j$ such that $\pi_j^k \circ \pi_k = \pi_j$ for each $k \ge j$ in Λ (see also § 2.5 in [Eng86]). Each element $x \in X$ is a thread $x = \{x_j : x_j \in X_j$ for each $j \in \Lambda$, $\pi_j^k(x_k) = x_j$ for each $k \ge j \in \Lambda\}$. Then $\pi_j^{-1}(\mathcal{R}_j) =: \mathcal{G}_j$ is the ring of subsets in X such that $\mathcal{G}_j \subset \tau_X$ for each $j \in \Lambda$. The base of topology of (X, τ_X) is formed by subsets $\pi_j^{-1}(A)$, where $A \in \tau_j$, $j \in \Lambda$, but $\mathcal{R}_j$ is the base of topology τ_j for each j, hence $\{B : B = \pi_j^{-1}(A), A \in \mathcal{R}_j,\ j \in \Lambda\}$ is the base of τ_X. Therefore, the ring $\mathcal{R} := \bigcup_{j\in\Lambda} \mathcal{G}_j$ is the base of τ_X. In view of Proposition 2.34 we can reduce our consideration to the case, when each $\mathcal{R}_j$ is separating on X_j and $\mathcal{R}$ is separating on X, since $\mathcal{G}_j \subset \mathcal{G}_k$ for each $k \ge j$ in Λ.

Consider on $\mathcal{R}$ a function μ with values in $\mathbf{K}$ such that $\mu(\pi_j^{-1}(A)) := \mu_j(A)$ for each $A \in \mathcal{R}_j$ and each $j \in \Lambda$. If A and B are disjoint elements in $\mathcal{R}$, then there exists $j \in \Lambda$ such that A and B are in $\mathcal{G}_j$, hence

(i) $A = \pi_j^{-1}(C)$ and $B = \pi_j^{-1}(D)$ for some C and D in $\mathcal{R}_j$, consequently, $\mu(A \cup B) = \mu_j(C \cup D) = \mu_j(C) + \mu_j(D) = \mu_j(A) + \mu_j(B)$, that is, μ is additive. Moreover, $\|A\|_\mu = \|C\|_{\mu_j}$ for each $A = \pi_j^{-1}(C)$ with $C \in \mathcal{R}_j$, hence $\|X\|_\mu = 1$. Since $\mu(X) = \mu_j(X_j)$ and $\mu_j(X_j) = 1$ for each $j \in \Lambda$, then $\mu(X) = 1$. Therefore, μ satisfies Conditions 2.1.(i, ii). It remains to verify Condition 2.1.(iii). By formula of § 2.1 we have the function $N_\mu(x)$ on $(X, \mathcal{R})$ such that for each $x \in X$ and $\varepsilon > 0$ there exists $A \in \mathcal{R}$ such that

(ii) $\|A\|_\mu - \varepsilon < N_\mu(x) \leq \|A\|_\mu$. In view of (i) and upper semi-continuity of $N_{\mu_j}(x_j)$ on $(X_j, \mathcal{R}_j)$ for each $x \in X$ and $\varepsilon > 0$ there exists $j \in \Lambda$ and its neighborhood $A = \pi_j^{-1}(C) \in \mathcal{R}$ such that

(iii) $N_{\mu_j}(y_j) < N_\mu(x) + \varepsilon$ for each $y \in A$, where $y_j := \pi_j(y)$. Hence for each $x \in X$ and each $\varepsilon > 0$ there exists a basic neighborhood A of x such that

(iv) $N_\mu(y) < N_\mu(x) + \varepsilon$ for each $y \in A$, that is, $N_\mu(x)$ is upper semi-continuous on $(X, \mathcal{R})$, since $0 \leq N_{\mu_j}(x_j) \leq 1$ for each $x_j \in X_j$ and $j \in \Lambda$. From Formulas (i, ii, iii) and 2.1.(ii) we have

(v) $\|A\|_\mu = \sup_{x \in X} N_\mu(x)$ for each $A \in \mathcal{R}$, since $\|A\|_\mu = \sup_{x \in C} N_{\mu_j}(x)$. For a compact subset V in X and each $\varepsilon > 0$ there exists a finite covering $\{E_1, \dots, E_m\} \subset \mathcal{R}$ of V such that inequalities $(ii - iv)$ are satisfied for each E_l instead of A. Therefore,

(vi) $\sup_{x \in V} N_\mu(x) \leq \max_{l=1,\dots,m} \|E_l\|_\mu \leq \sup_{x \in V} N_\mu(x) + 2\varepsilon$ and

(vii) $\sup_{x \in V} N_\mu(x) = \inf_{\mathcal{R} \ni A \supset V} \|A\|_\mu$. Though the compact subset V is not necessarily in $\mathcal{R}$ we take Equation (vii) as the definition of $\|V\|_\mu := \inf_{\mathcal{R} \ni A \supset V} \|A\|_\mu$.

Choose a sequence $\varepsilon_j = \delta > 0$ for each $j \in \Lambda$, where $\delta > 0$ is independent from j. For each $\varepsilon_j > 0$ a subset $X_{j,\varepsilon_j} := \{x_j : x_j \in X_j, N_{\mu_j}(x_j) \geq \varepsilon_j > 0\}$ is compact. If $x_k \in X_{k,\varepsilon_k}$, then $N_{\mu_j}(\pi_j^k(x_k)) \geq \varepsilon_k$ for each $j < k$, since $(\pi_j^k)^{-1}(\mathcal{R}_j) \subset \mathcal{R}_k$ and $\|B\|_{\mu_k} \leq \|A\|_{\mu_k}$ for each B and A in $\mathcal{R}_k$ with $B \subset A$. Hence $\pi_j^k(X_{k,\varepsilon_k}) \subset X_{j,\varepsilon_k}$ for each $j \leq k$ in Λ. Since $\pi_k^m \circ \pi_l^k = \pi_l^m$ for each $m \geq k \geq l$ in Λ, then $\{X_{k,\delta}, \pi_j^k, \Lambda\}$ is the inverse mapping system. The image $\pi_j^k(X_{k,\delta})$ of each compact set $X_{k,\delta}$ is compact for each $k > j$ (see also Theorem 3.1.10 [Eng86]), since each (X_k, τ_k) is the Hausdorff space in our consideration. Since the limit of an inverse mapping system of compact spaces is compact (see also Theorem 3.2.13 [Eng86]), then the limit $X_{\{\varepsilon_j : j\}} := \lim\{X_{k,\varepsilon_k}, \pi_j^k, \Lambda\}$ is the compact subset in X such that $X_{\{\varepsilon_j : j\}}$ is homeomorphic with $\theta(X) \cap \prod_{k \in \Lambda} X_{k,\varepsilon_k}$, where $\theta : X \hookrightarrow \prod_{k \in \Lambda} X_k$ is the embedding.

For a shrinking family $\mathcal{S}$ in $\mathcal{R}$ consider all finite intersections of finite families in $\mathcal{S}$, this gives a centered family $\mathcal{S}_0$ in $\mathcal{R}$ such that $\mathcal{S} \subset \mathcal{S}_0$ and denote it also by $\mathcal{S}$. Applying $(i - vii)$ to $V = X_{\{\varepsilon_j : j\}}$ and using basic neighborhoods $U = \pi_k^{-1}(U_k)$, where $U_k \in \mathcal{R}_k$, we get that for each shrinking family $\mathcal{S}$ in $\mathcal{R}$ with $\bigcap \mathcal{S} = \emptyset$ there exists $\lim_{A \in \mathcal{S}} \|A\|_\mu = 0$, since due to (vi) we have

$(viii)$ $N_\mu(x) \leq \delta$ for each $x \in X \setminus X_{\{\varepsilon_j : j\}}$, since $N_{\mu_j}(x_j) < \delta$ for each $x_j \in X \setminus X_{j,\delta}$ and each $j \in \Lambda$. Using the completion of $\mathcal{R}$ relative to μ we get the probability measure space $(X, \mathcal{R}_\mu, \mu)$.

2.38. Note. Theorem 2.37 has an evident generalization to the following case: $\|X_j\|_{\mu_j} < \infty$ for each j and there exist two limits $\lim_{j \in \Lambda_0} \mu_j(X_j) \in \mathbf{K}$ and $\lim_{j \in \Lambda} \|X_j\|_{\mu_j} < \infty$, where $\Lambda_0 := \{j : j \in \Lambda, \mu_j(X_j) \neq 0\}$ and $\Lambda \setminus \Lambda_0$ is bounded in Λ. We have $\|X_j\|_{\mu_j} \leq \|X_k\|_{\mu_k}$ for each $j \leq k$ in Λ, since $\pi_j^k(\mu_k) = \mu_j$ and $(\pi_j^k)^{-1}(\mathcal{R}_k) \subset \mathcal{R}_j$. Since Λ is directed, then $\lim_{j \in \Lambda} \|X_j\|_{\mu_j} = \sup_{j \in \Lambda} \|X_j\|_{\mu_j}$. Since a cylindrical distribution μ is defined on $\mathcal{R}$

and bounded on it, then μ has an extension to the bounded measure μ on $\mathcal{R}_\mu$ such that $\mu(X) = \lim_{j\in\Lambda}\mu_j(X_j)$ and $\|X\|_\mu = \lim_{j\in\Lambda}\|X_j\|_{\mu_j}$, where $\mathcal{R}_\mu$ is the completion of $\mathcal{R}$ relative to μ.

Let now X be a set with a covering ring $\mathcal{R}$ such that $X \in \mathcal{R}$. Let also $\{(X,\mathcal{G}_j,\mu_j) : j \in \Lambda\}$ be a family of measure spaces such that Λ is directed and $\mathcal{G}_j \subset \mathcal{G}_k$ for each $j \le k \in \Lambda$, $\mathcal{R} = \bigcup_{j\in\Lambda}\mathcal{G}_j$. Suppose $\mu : \mathcal{R} \to \mathbf{K}$ is such that $\mu|_{\mathcal{G}_j} = \mu_j$ and $\mu_k|_{\mathcal{G}_j} = \mu_j$ for each $j \le k$ in Λ. Then the triple $(X,\mathcal{R},\mu)$ is called the cylindrical distribution. For each $A \in \mathcal{R}$ there exists $j \in \Lambda$ such that $A \in \mathcal{G}_j$, hence $\|A\|_{\mu_j} = \|A\|_{\mu_k}$ for each $k \ge j$ in Λ, consequently, $\|A\|_\mu := \lim_{k\in\Lambda}\|A\|_{\mu_k}$ is correctly defined. Suppose μ is bounded, that is,$\|X\|_\mu < \infty$. (A particular simpler case is given below in § 2.41).

2.39. Theorem. *Let $(X,\mathcal{R},\mu)$ be a bounded cylindrical distribution as in* § 2.38. *Then μ has an extension to a bounded measure μ on a completion $\mathcal{R}_\mu$ of $\mathcal{R}$ relative to μ.*

Proof. Let τ_X be a topology on X generated by the base $\mathcal{R}$. In view of Proposition 2.34 each covering ring $\mathcal{G}_j$ of X produces an equivalence relation κ_j and a quotient mapping $\pi_j : X \to X_j$ such that $\pi_j(\mathcal{G}_j) =: \mathcal{R}_j$ is a separating covering ring of X_j, where X_j is zero-dimensional and Hausdorff. Moreover, $\mathcal{R}_j$ is a base of a topology τ_j on X_j. Since $\mathcal{G}_k \supset \mathcal{G}_j$ for each $k \ge j$, then on $(X_k,(\pi^k_j)^{-1}(\mathcal{R}_j))$ there exists an equivalence relation κ^k_j and a quotient (continuous) mapping $\pi^k_j : X_k \to X_j$ such that $\pi^m_k \circ \pi^k_j = \pi^m_j$ for each $j \le k \le m$ in Λ. Hence there exists an inverse mapping system $\{X_k,\pi^k_j,\Lambda\}$. Therefore, the set X in the topology τ_X generated by its base $\mathcal{R}$ consisting of clopen subsets is homeomorphic with $\lim\{X_k,\pi^k_j,\Lambda\}$. Each $\pi_j(\mu) = \mu_j$ is a bounded measure on $(X_j,\mathcal{R}_j)$ such that $\pi^k_j(\mu_k) = \mu_j$ and $(\pi^k_j)^{-1}(\mathcal{R}_j) \subset \mathcal{R}_k$ for each $k \ge j \in \Lambda$. Therefore, $\{(X_j,\mathcal{R}_j,\mu_j) : j \in \Lambda\}$ is the consistent family of measure spaces. From the definition of μ it follows that μ is additive, hence $\|X\|_\mu$ is correctly defined. From $\|X\|_\mu < \infty$ it follows $\|X_j\|_{\mu_j} < \infty$ for each $j \in \Lambda$ and there exists $\lim_{j\in\Lambda}\|X_j\|_{\mu_j} = \|X\|_\mu$. From $X \in \mathcal{R}$ it follows, that $\mu(X) = \mu_j(X_j)$ for each $j \in \Lambda$. Then this Theorem follows from Theorem 2.37.

2.40. Note. Let $X := \prod_{t\in T} X_t$ be a product of sets X_t and on X a covering ring $\mathcal{R}$ be given such that for each $n \in \mathbf{N}$ and pairwise distinct points $t_1,\dots,t_n$ in a set T there exists a $\mathbf{K}$-valued measure $P_{t_1,\dots,t_n}$ on a covering ring $\mathcal{R}_{t_1,\dots,t_n}$ of $X_{t_1} \times \dots \times X_{t_n}$ such that $\pi^{t_1,\dots,t_{n+1}}_{t_1,\dots,t_n}(\mathcal{R}_{t_1,\dots,t_{n+1}}) = \mathcal{R}_{t_1,\dots,t_n}$ for each $t_{n+1} \in T$ and $P_{t_1,\dots,t_{n+1}}(A_1 \times \dots \times A_n \times X_{n+1}) = P_{t_1,\dots,t_n}(A_1 \times \dots \times A_n)$ for each $A_1 \times \dots \times A_n \in \mathcal{R}_{t_1,\dots,t_n}$, where $\pi^{t_1,\dots,t_{n+1}}_{t_1,\dots,t_n} : X_{t_1} \times \dots \times X_{t_{n+1}} \to X_{t_1} \times \dots \times X_{t_n}$ is the natural projection, $A_l \subset X_{t_l}$ for each $l = 1,\dots,n$. Suppose that this cylindrical distribution is bounded, that is,

$\sup_{t_1,\dots,t_n\in T,n\in\mathbf{N}}\|P_{t_1,\dots,t_n}\| < \infty$ and there exists

$\lim_{t_1,\dots,t_n\in T_0;n\in\mathbf{N}} P_{t_1,\dots,t_n}(X_{t_1} \times \dots \times X_{t_n}) \in \mathbf{K}$, where $T_0 := \{t \in T : P_t(X_t) \ne 0\}$, $T \setminus T_0$ is finite.

2.41. Theorem (the non-Archimedean analog of the Kolmogorov theorem)**.** *A cylindrical distribution $P_{t_1,\dots,t_n}$ from* § 2.40 *has an extension to a bounded measure P on a completion $\mathcal{R}_P$ of $\mathcal{R} := \bigcup_{t_1,\dots,t_n\in T,n\in\mathbf{N}}\mathcal{G}_{t_1,\dots,t_n}$ relative to P, where $\mathcal{G}_{t_1,\dots,t_n} := (\pi_{t_1,\dots,t_n})^{-1}(\mathcal{R}_{t_1,\dots,t_n})$ and $\pi_{t_1,\dots,t_n} : X \to X_{t_1} \times \dots \times X_{t_n}$ is the natural projection.*

2.3. Quasi-invariant $\mathbf{K_s}$-Valued Measures

In this section after few preliminary statements there are given the definition of a quasi-invariant measure and the theorems about quasi-invariance of measures relative to transformations of a Banach space X over $\mathbf{K}$.

3.1. Let X be a Banach space over $\mathbf{K}$, $(L_n : n)$ be a sequence of subspaces, $cl(\bigcup_n L_n) = X$, $L_n \subset L_{n+1}$ for each n, μ^j be probability measures, $\mu^2 \ll \mu^1$, $(\mu^j_{L_n})$ be sequences of weak distributions, also let there exist derivatives $\rho_n(x) = \mu^2_{L_n}(dx)/\mu^1_{L_n}(dx)$ and the following limit $\rho(x) := \lim_{n\to\infty} \rho_n(x)$ exists.

Theorem. *If μ^j are $\mathbf{K_s}$-valued and $[\rho_n(P_{L_n}x) : n]$ converges uniformly on $Af(X,\mu^1)$-compact subsets in X, $\sup_n \|\rho_n\|_{N_{\mu^1}} < \infty$, then this is equivalent to the following: $\rho(x) = \mu^2(dx)/\mu^1(dx) \in L(\mu^1)$ and $\lim_{n\to\infty} \|\rho(x) - \rho_n(P_{L_n}x)\|_{N_{\mu^1}} = 0$.*

Proof. For each $A \in Bco(L)$ the equality is accomplished:

$$\mu^2_L(A) = \int_A \rho_L(x)\mu^1_L(dx) = \int_{P_L^{-1}(A)} \rho_L(P_L x)\mu^1(dx).$$

Then for each $\psi \in L(\mu^1|P_L^{-1}[Bco(L)])$ we have

$$\int_X \psi(x)\mu^2(dx) = \int_X \rho_L(P_L x)\psi(x)\mu^1(dx), \text{ consequently,}$$

$$\int_X \rho_{n+1}(x)\psi(x)\mu^1(dx) = \int_X \psi(x)\mu^2(dx) = \int_X \rho_n(x)\psi(x)\mu^1(dx),$$

where $\rho_{L_n} = \rho_n$, $\psi \in L(\mu^1|P^{-1}_{L_{n+1}}[Bco(L_{n+1})])$. From Theorem 2.33 and Definition 2.31 the statement follows.

3.2. Theorem. (A). *Measures $\mu^j : E \to \mathbf{K_s}$, $j = 1,2$, for a Banach space X over $\mathbf{K}$ are orthogonal $\mu^1 \perp \mu^2$ if and only if $N_{\mu^1}(x)N_{\mu^2}(x) = 0$ for each $x \in X$.*

(B). *If for measures $\mu^j : E \to \mathbf{K_s}$ on a Banach space X over $\mathbf{K}$ is satisfied $\rho(x) = 0$ for each x with $N_{\mu^1}(x) > 0$, then $\mu^1 \perp \mu^2$; the same is true for a completely regular space X with $ind(X) = 0$ and $\rho(x) = \mu^2(dx)/\mu^1(dx) = 0$ for each x with $N_{\mu^1}(x) > 0$.*

Proof. (A). From Definition 2.31 it follows that there exists $F \in E$ with $\|X \setminus F\|_{\mu^1} = 0$ and $\|F\|_{\mu^2} = 0$.

Remind Theorem 7.20 about tight measures from [Roo78]. For a function $f : X \to \mathbf{K}$ the conditions (α) and (β) are equivalent.

(α). For every $\mu \in M(X)$ the function f is μ-integrable.

(β). The function f is bounded and for each compact subset C in X the restriction of f to C is continuous.

In view of Theorems 7.6 [Roo78] recalled above and the Theorem just cited above the characteristic function ch_F of the set F belongs to $L(\mu^1) \cap L(\mu^2)$ such that $N_{\mu^j}(x)$ are semi-continuous from above, $\|ch_F\|_{N_{\mu^2}} = 0$, $\|ch_{X\setminus F}\|_{N_{\mu^1}} = 0$, consequently, $N_{\mu^1}(x)N_{\mu^2(x)} = 0$ for each $x \in X$.

Recall Lemma 7.2 [Roo78]. Let μ be a measure on $\mathcal{R}$. There exists a unique function $N_\mu : X \to [0,\infty)$ for which

(Ri) $\|\xi_U\|_{N_\mu} = \|U\|_\mu$ for each $U \in \mathcal{R}$, where ξ_U denotes the characteristic function of U;

(Rii) if $\phi : X \to [0,\infty)$ and $\|\xi_U\|_\phi \le \|U\|_\mu$ for all $U \in \mathcal{R}$, then $\phi \le N_\mu$, moreover, N_μ is given by the formula $N_\mu(x) = \inf_{U\in\mathcal{R},x\in U} \|U\|_\mu$.

On the other hand, if $N_{\mu^1}(x)N_{\mu^2}(x) = 0$ for each x, then for $F := [x \in X : N_{\mu^2}(x) = 0]$ due to the lemma recalled just above $\|F\|_{\mu^2} = \|ch_F\|_{N_{\mu^2}} = 0$. Moreover, in view of Theorem 7.6[Roo78] reminded above $F = \bigcap_{n=1}^\infty U_{s^{-n}}$, where $U_c := [x \in X : N_{\mu^2}(x) < c]$ are open in X, hence $ch_F \in L(\mu^1) \cap L(\mu^2)$ and $N_{\mu^1}|_{(X\setminus F)} = 0$, consequently, $\|X \setminus F\|_{\mu^1} = 0$.

(B). In view of Theorems 2.19 and 2.37 for each $A \in P_{L_n}^{-1}[E(L_n)]$ and $m > n$: $\int_A \rho_m(x)\mu^1(dx) = \mu^2(A)$, then from $\lim_{n\to\infty} \|\rho(x) - \rho_n(P_{L_n}x)\|_{N_{\mu^1}} = 0$ and Conditions 2.1.(i-iii) on μ^2 Statement (B) follows.

3.3. Note. The Radon-Nikodym theorem is not valid for μ^j with values in $\mathbf{K_s}$, so not all theorems for real-valued measures may be transferred onto this case. Therefore, the definition of absolute continuity of measures was changed (see § 2.31 above and [Sch71]).

3.4. Theorem. *Let measures μ^j and ν^j be with values in $\mathbf{K_s}$ on $Bco(X_j)$ for a Banach space X_j over $\mathbf{K}$ and $\mu = \mu^1 \otimes \mu^2$, $\nu = \nu^1 \otimes \nu^2$ on $X = X_1 \otimes X_2$, therefore, the statement $\nu \ll \mu$ is equivalent to $\nu^1 \ll \mu^1$ and $\nu^2 \ll \mu^2$, moreover, $\nu(dx)/\mu(dx) = (\nu^1(P_1dx)/\mu^1(P_1dx))(\nu^2(P_2dx)/\mu^2(P_2dx))$, where $P_j : X \to X_j$ are projectors.*

Proof. Recall the non-Archimedean analog of the Fubini theorem (see also Theorem 7.15 [Roo78]). Let μ and ν be $\mathbf{K}$-valued measures on separating covering rings $\mathcal{R}$ and $\mathcal{S}$ on X and Y correspondingly. The finite unions of sets $A \times B$ for $A \in \mathcal{R}$ and $B \in \mathcal{S}$ form a covering ring $\mathcal{R} \otimes \mathcal{S}$ of $X \times Y$. Moreover,

(Ri) there exists a unique measure $\mu \times \nu$ on $\mathcal{R} \otimes \mathcal{S}$ so that $(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$, $N_{\mu\times\nu}(x,y) = N_\mu(x)N_\nu(y)$;

(Rii) if $f \in L(\mu)$, $g \in L(\nu)$, then $f \otimes g \in L(\mu \times \nu)$ and $\int f \otimes g d(\mu \times \nu) = \int f d\mu \int g d\nu$;

$(Riii)$ if $f \in L(\mu \times \nu)$, then $y \mapsto \int f(x,y)d\mu(x)$ is ν-almost everywhere defined ν-integrable function and $\int f d(\mu \times \nu) = \int \int f(x,y)d\mu(x)d\nu(y)$.

This theorem follows from the formulated just above non-Archimedean analog of the Fubini theorem and making the modification of the proof of Theorem 3.2.3 of Chapter I above.

3.5. Theorem. The non-Archimedean analog of the Kakutani theorem. *Let $X = \prod_{j=1}^\infty X_j$ be a product of completely regular spaces X_j with $ind(X_j) = 0$ and probability measures $\mu^j, \nu^j : E(X_j) \to \mathbf{K_s}$, also let $\mu_j \ll \nu_j$ for each j, $\nu = \bigotimes_{j=1}^\infty \nu_j$, $\mu = \bigotimes_{j=1}^\infty \mu_j$ are measures on $E(X)$, $\rho_j(x) = \mu_j(dx)/\nu_j(dx)$ are continuous by $x \in X_j$, $\prod_{j=1}^n \rho_j(x_j) =: t_n(x)$ converges uniformly on $A f(X,\mu)$-compact subsets in X, $\beta_j := \|\rho_j(x)\|_{\phi_j}$, $\phi_j(x) := N_{\nu^j}(x)$ on X_j. If $\prod_{j=1}^\infty \beta_j$ converges in $(0,\infty)$ (or diverges to 0), then $\mu \ll \nu$ and $q_n(x) = \prod_{j=1}^n \rho_j(x_j)$ converges in $L(X,\nu,\mathbf{K_s})$ to $q(x) = \prod_{j=1}^\infty \rho_j(x_j) = \mu(dx)/\nu(dx)$ (or $\mu \perp \nu$ respectively), where $x_j \in X_j$, $x \in X$.*

Proof. The countable additivity of ν and μ follows from Theorem 2.19. Then $\beta_j = \|\rho_j\|_{\phi_j} \le \|\rho_j\|_{N_{\nu_j}} = \|X\|_{\mu_j} = 1$, since $N_{\nu_j} \le 1$ for each $x \in X_j$, hence $\prod_{j=1}^\infty \beta_j$ can not be divergent to ∞. If this product diverges to 0 then there exists a sequence $\varepsilon_b := \prod_{j=n(b)}^{m(b)} \beta_j$ for which the series converges $\sum_{b=1}^\infty \varepsilon_b < \infty$, where $n(b) \le m(b)$. For $A_b := [x : (\prod_{j=n(b)}^{m(b)} \rho_j(x_j)) \ge 1]$ there are estimates $\|A_b\|_\nu \le \sup_{x\in A_b}[\prod_{j=n(b)}^{m(b)} |\rho_j(x_j)|\phi_j(x_j)] \le \varepsilon_b$, consequently, $\|A\|_\nu = 0$ for $A = \limsup(A_b : b \to \infty)$, since $0 < \sum_{b=1}^\infty \varepsilon_b < \infty$.

For $B_b := X \setminus A_b$ we have:

$$\|B_b\|_\mu \leq \left[\sup_{x\in B_b} \prod_{j=n(b)}^{m(b)} |1/\rho_j(x_j)|\psi(x_j)\right] = \left[\prod_{j=n(b)}^{m(b)} \|\rho_j(x_j)\|_{\phi_j}\right] = \varepsilon_b$$

, where $\psi_j(x) = N_{\mu_j}(x)$, since $\mu_j(dx_j) = \rho_j(x_j)\nu_j(dx_j)$ and $N_{\mu_j}(x) = |\rho_j(x_j)|N_{\nu_j}(x)$ due to continuity of $\rho_j(x_j)$ (for $\rho_j(x_j) = 0$ we set $|1/\rho_j(x_j)|\psi_j(x_j) = 0$, because $\psi_j(x_j) = 0$ for such x_j), consequently, $\|\limsup(B_b : b \to \infty)\|_\mu = 0$ and $\|A\|_\mu \geq \|\liminf(A_b : b \to \infty)\|_\mu = 1$. This means that $\mu \perp \nu$.

Suppose that $\prod_{j=1}^\infty \beta_j$ converges to $0 < \beta < \infty$, then $\beta \leq 1$ (see above). Therefore from the non-Archimedean analog of the Lebesgue reminded above in this Chapter it follows that $t_n(x)$ converges in $L(X,\mu,\mathbf{K_s})$, since $|t_n(x)| \leq 1$ for each x and n, at the same time each $t_n(x)$ converges uniformly on compact subsets in the topology generated by $Af(X,\mu)$. Then for each bounded continuous cylindrical function $f : X \to \mathbf{K_s}$ we have

$$\int_X f(x)\mu(dx) = \int_X f(x_1,\dots,x_n)t_n(x) \otimes_{j=1}^n \nu_j(dx_j)$$

$$= \lim_{n\to\infty} \int_X f(x)t_n(x)\nu(dx) = \int_X \rho(x)\nu(dx).$$

Approximating arbitrary $h \in L(X,\mu,\mathbf{K_s})$ by such f we get the equality

$$\int_X h(x)\mu(dx) = \int_X h(x)\rho(x)\nu(dx),$$

consequently, $\rho(x) = \mu(dx)/\nu(dx)$.

3.6. Theorem. *Let* ν, μ, ν_j, μ_j *be probability measures with values in* $\mathbf{K_s}$, X *and* X_j *be the same as in* § 3.5 *and* $\mu \ll \nu$, *then* $\mu_j \ll \nu_j$ *for each* j *and* $\prod_{j=1}^\infty \beta_j$ *converges to* β, $\infty > \beta > 0$, *where* $\beta_j = \|\rho_j\|_{\phi_j}$, $\phi_j(x) = N_{\nu_j}(x)$.

Proof. For $\mathbf{K_s}$-valued measures from $P_j^{-1}(Bco(X_j)) \subset Bco(X)$ it follows that $\mu_j \ll \nu_j$ for each j, since $\prod_1^\infty \rho_j(x_j) = \rho(x) \in L(X,\nu)$ and $\rho_j(x_j) \in L(X_j,\nu_j)$, where $x_j = P_j x$, $P_j : X \to X_j$ are projectors. Then $\rho(x) = \lim_{n\to\infty} \prod_1^n \rho_j(P_j x)$ and $\|\rho(x)\|_{N_\nu} = \lim_{n\to\infty} \|\rho_j\|_{N_{\nu_j}}$. Since $N_{\nu_j} \leq 1$, then $\phi_j(x) \leq N_{\nu_j}(x)$ and for $\phi = N_\nu$, consequently, $\|\rho(x)\|_\phi = \lim_{n\to\infty} \prod_{j=1}^n \|\rho_j\|_{\phi_j} \leq \|\rho\|_{N_\nu} = 1$ (due to the definition of the Tihonov topology in X [see also § 2.3[Eng86]] and definition of $\|*\|_\phi$). If $\|\rho\|_\phi = 0$, then $\|\rho\|_{N_\nu} = 0$ and by Theorem 3.2(B) this would mean that $\nu \perp \mu$ or $\mu = 0$, but $\mu \neq 0$, hence $\beta > 0$.

3.7. Definition. Let X be a Banach space over $\mathbf{K}$, Y be a completely regular space with $ind(X) = 0$, $\nu : Bco(Y) \to \mathbf{K_s}$, $\mu^y : Bco(X) \to \mathbf{K_s}$ for each $y \in Y$, suppose $\mu^y(A) \in L(Y,\nu)$ for each $A \in Bco(X)$, $\|Y\|_\nu < \infty$, $\sup_{y\in Y} \|X\|_{\mu^y} < \infty$, a family $(\mu^y(A_n) : n)$ is converging uniformly by $y \in C$ on each $Af(Y,\nu)$-compact subset C in Y for each given shrinking family of subsets $(A_n : n) \subset X$. Then we define:

$$(i)\ \mu(A) = \int_Y \mu^y(A)\nu(dy).$$

A measure μ is called mixed. Evidently, Condition 2.1(i) is fulfilled; (ii): $\|A\|_\mu \leq (\sup_{y\in Y} \|A\|_{\mu^y})\|A\|_\nu < \infty$; (iii) is carried out due to the non-Archimedean analog of the Lebesgue theorem, since

$$\lim_{n\to\infty} \mu(A_n) = \int_Y (\lim_n \mu^y(A_n))\nu(dy) = 0.$$

We define measures π^j by the formula:

$$(ii)\ \pi^j(A\times C)=\int_C \mu^{j,y}(A)\nu^j(dy),$$

where $j=1,2$ and $\mu^{y,j}$ together with ν^j are defined as above μ^y and ν.

3.8. Theorem. *Let μ^j be $\mathbf{K_s}$-valued measures and π^j, X and Y be the same as in* § 3.7.

(A). *If $\pi^2\ll\pi^1$, then $\nu^2\ll\nu^1$ and $\mu^{2,y}\ll\mu^{1,y}$ $(mod\ \nu^2)$.*

(B). *If $\nu^2\ll\nu^1$ and $\mu^{2,y}\ll\mu^{1,y}$ $(mod\ \nu^2)$ and a $Bco(X\times Y,\pi^1)$-measurable function $\tilde\rho(y,x)=\mu^{2,y}(dx)/\mu^{1,y}(dx)\in L(X\times Y,\pi^1)$ exists, then $\pi^2\ll\pi^1$ and $\pi^2(d(x,y))/\pi^1(d(x,y))=(\nu^2(dy)/\nu^1(dy))\tilde\rho(y,x)$.*

Proof. (A). From the conditions imposed on $\mu^{j,y}$ and ν^j it follows that for each $\phi\in L(X\times Y,\pi^j)$ due to the non-Archimedean analog of the Fubini theorem recalled above the following equality is accomplished

$$\int_{X\times Y}\phi(x,y)\pi^j(d(x,y))=\int_Y\left[\int_X\phi(x,y)\mu^{j,y}(dx)\right]\nu^j(dy),$$

also $\rho(y,x)=\pi^2(d(x,y))/\pi^1(d(x,y))\in L(X\times Y,\pi^1)$, hence

$$\nu^2(dy)/\nu^1(dy)=\left[\int_X\rho(y,x)\mu^{1,y}(dx)\right]\in L(Y,\nu^1).$$

Further we modify the proof of Theorem I.3.1 above. Then $\tilde\rho(y,x)$ may be defined for ν^2-almost all y by $\tilde\rho(y,x)=\rho(y,x)/\int_X\rho(y,x)\mu^{1,y}(dx)\in L(X,\mu^{1,y})$.

(B). Let $A\in Bco(X)\times Bco(Y)$, $A_y:=[y:(x,y)\in A]$, then $\pi^j(A)=\int_Y\mu^{j,y}(A_y)\nu^j(dy)$. If $\|A\|_{\pi^1}=0$, then $\|A_y\|_{\mu^{1,y}}N_{\nu^1}(y)=0$ for each $y\in Y$, consequently, $\|A\|_{\pi^2}=0$, since $\nu^2(dy)/\nu^1(dy)\in L(\nu^1)$, $\mu^{2,y}(dx)/\mu^{1,y}(dx)\in L(\mu^{1,y})$, $\tilde\rho\in L(X\times Y,\pi^1)$ and Conditions (i,ii) in § 3.7 are satisfied. From this it follows that $\pi^2(d(x,y))/\pi^1(d(x,y))\in L(X\times Y,\pi^1)$, since $\nu^2(dy)/\nu^1(dy)\in L(X\times Y,\pi^1)$ with $\sup_y\|X\|_{\mu^{j,y}}<\infty$.

3.9. Definition. For a Banach space X over $\mathbf{K}$ an element $a\in X$ is called an admissible shift of a measure μ with values in $\mathbf{K_s}$, if $\mu_a\ll\mu$, where $\mu_a(A)=\mu(S_{-a}A)$ for each A in $E\supset Bco(X)$, $S_aA:=a+A$, $\rho(a,x):=\rho_\mu(a,x):=\mu_a(dx)/\mu(dx)$, $M_\mu:=[a\in X:\mu_a\ll\mu]$ (see § § 2.1 and 2.31).

3.10. Properties of M_μ and ρ from § 3.9.

I. *The set M_μ is a semigroup by addition, $\rho(a+b,x)=\rho(a,x)\rho(b,x-a)$ for each $a,b\in M_\mu$.*

Proof. For each continuous bounded $f:X\to\mathbf{K_s}$: $\int_X f(x)\mu_{a+b}(dx)=\int_X f(x+a+b)\mu(dx)=\int_X f(x+a)\rho(b,x)\mu(dx)=\int_X f(x)\rho(b,x-a)\rho(a,x)\mu(dx)$, since $\|X\|_\mu<\infty$ and $f(x)\rho(b,x-a)\in L(\mu)$, consequently, $\rho(b,x-a)\rho(a,x)=\rho(a+b,x)\in L(\mu)$ as a function of x and $\mu_{a+b}\ll\mu$.

II. *If $a\in M_\mu$, $\rho(a,x)\ne0$ $(mod\ \mu)$, then $\mu_a\sim\mu$, $-a\in M_\mu$ and $\rho(-a,x)=1/\rho(a,x-a)$.*

Proof. For each continuous bounded $f:X\to\mathbf{K_s}$: $\int_{\mathbf{X}}\mathbf{f}(\mathbf{x})\mu(\mathbf{dx})=\int_{\mathbf{X}}\mathbf{f}(\mathbf{x})\mu(\mathbf{dx})=\int_{\mathbf{X}}[\mu_{\mathbf{a}}(\mathbf{dx})/\mu(\mathbf{dx})]^{-\mathbf{1}}\mu_{\mathbf{a}}(\mathbf{dx})$, since $\|X\|_\mu<\infty$, hence $\mu_a\sim\mu$.

III. *If $\nu\ll\mu$ and $\nu(dx)/\mu(dx)=g(x)$, then $M_\mu\cap M_\nu=M_\mu\cap[a:\mu([x:g(x)=0,g(x-a)\rho_\mu(a,x)\ne0])=0]$ and $\rho_\nu(a,x)=[g(x-a)/g(x)]\rho_\mu(a,x)$ $(mod\ \nu)$ for $a\in M_\mu\cap M_\nu$.*

Proof. For each continuous bounded function $f : X \to \mathbf{K_s}$: $a \in M_\mu$ and $\int_X f(x+a)\nu(dx) = \int_X f(x)g(x-a)\rho_\mu(a,x)\mu(dx)$ such that $\mu([x : g(x) = 0, g(x-a)\rho_\mu(a,x) \neq 0]) = 0$ we have $\int_X f(x+a)\nu(dx) = \int_X f(x)[g(x-a)\rho_\mu(a,x)/g(x)]\nu(dx)$, since $\|X\|_\nu + \|X\|_\mu < \infty$, $N_\nu(x) = \inf_{Bco(X)\supset U \ni x} \sup_{y\in U}[\ |g(y)|N_\mu(y)]$, consequently, $a \in M_\mu \cap M_\nu$. If $a \in M_\mu \cap M_\nu$, then

$$\int_X f(x)\rho_\nu(a,x)g(x)\mu(dx) = \int_X f(x)g(x-a)\rho_\mu(a,x)\mu(dx),$$

consequently, $\rho_\nu(a,x)g(x) = g(x-a)\rho_\mu(a,x) \ (mod\ \mu)$ and $\mu([x : g(x) = 0, g(x-a)\rho_\mu(a,x) \neq 0]) = 0$.

IV. *If* $\nu \sim \mu$, *then* $M_\nu = M_\mu$.

V. *For* μ *with values in* $\mathbf{K_s}$ *and* $X = K^m$, $m \in \mathbf{N}$ *a family* M_μ *with a distance function* $r(a,b) = \|\rho(a,x) - \rho(b,x)\|_{N_{\mu(x)}}$ *is a complete pseudo-ultrametrizable space.*

Proof. Let $(a_n) \subset M_\mu$ be a Cauchy sequence relative to r, then (a_n) is bounded in X by $\|*\|_X$, since for $\lim_{j\to\infty}\|a_{n_j}\| = \infty$ and $r(a_{n_j}, a_{n_{j+1}}) < p^{-j}$ for $f \in L(\mu)$ with a compact support we have $\|f(x+a_{n_j}) - f(x+a_{n_1})\|_{N_\mu} < 1/p$. Then for f with $\|f(x+a_{n_1})\|_{N_\nu} > 1/2$ and $\|f\|_{N_\nu} = 1$ we get a contradiction: $\lim_j \|f(x+a_{n_j})\|_{N_\mu} > 1/2 - 1/p \geq 0$. This is impossible because of compactness of $supp(f)$. Therefore, (a_n) is bounded, consequently, there exists a subsequence $(a_{n_j}) =: (b_j)$ weakly converging in X to $b \in X$. Therefore, $\theta_j(z) = \int_X \chi_e(z(x+b_j))\mu(dx)\ \chi_e(z(b_j))\theta(z) = \int_X \chi_e(z(x))\rho(b_j,x)\mu(dx)$, $\lim_j z(b_j) = z(b)$ and $\lim_j \theta_j(z) = \chi_e(z(b))\theta(z)$ for each $z \in X'$.

From the basic theorem about the Fourier-Stieltjes transform reminded above in this chapter it follows that there is $\rho \in L(\mu)$ with $\lim_j \|\rho(b_j,x) - \rho(x)\|_{N_\mu} = 0$, since $L(\mu)$ is the Banach space and μ_j corresponding to θ_j converges in the Banach space $\mathsf{M}(X)$. Therefore,

$$\int_X \chi_e(z(x))\mu_b(dx) = \int_X \chi_e(z(x))\rho(x)\mu(dx)$$

for each $z \in X' = K^m$, consequently, $\rho(x) = \mu_b(dx)/\mu(dx)$.

3.11. Definition. For a Banach space X over $\mathbf{K}$ and a measure $\mu : Bco(X) \to \mathbf{K_s}$, $a \in X$, $\|a\| = 1$, a vector a is called an admissible direction, if $a \in M_\mu^{\mathbf{K}} := [z : \|z\|_X = 1$, $\lambda z \in M_\mu$ and $\rho(\lambda z, x) \neq 0\ (mod\ \mu)$ (relative to x) and for each $\lambda \in \mathbf{K}] \subset X$. Let $a \in M_\mu^{\mathbf{K}}$ we denote by $L_1 := \mathbf{K}a$, $X_1 = X \ominus L_1$, μ^1 and $\tilde{\mu}^1$ are the projections of μ onto L_1 and X_1 respectively, $\tilde{\mu} = \mu^1 \otimes \tilde{\mu}^1$ be a measure on $Bco(X)$, given by the the following equation $\tilde{\mu}(A \times C) = \mu^1(A)\tilde{\mu}^1(C)$ on $Bco(L_1) \times Bco(X_1)$ and extended on $Bco(X)$, where $A \in Bco(L_1)$ and $C \in Bco(X_1)$.

3.12. Definition and notes. A measure $\mu : Bco(X) \to \mathbf{K_s}$ for a Banach space X over $\mathbf{K}$ is called a quasi-invariant measure if M_μ contains a $\mathbf{K}$-linear manifold J_μ dense in X.

From § 3.10 and Definition 3.11 it follows that $J_\mu \subset M_\mu^{\mathbf{K}}$.

Let $(e_j : j \in \mathbf{N})$ be orthonormal basis in X, $H = span_{\mathbf{K}}(e_j : j)$. We denote $\Omega(Y) = [\mu|\mu$ is a measure with a finite total variation on $Bco(X)$ and $H \subset J_\mu]$, where $Y = \mathbf{K_s}$.

3.13. Theorem. *If* $\mu : Bf(Y) \to \mathbf{F}$ *is a* σ*-finite measure on* $Bco(Y)$, Y *is a complete separable ultra-metrizable* $\mathbf{K}$*-linear subspace such that* $co(S)$ *is nowhere dense in* Y *for each compact* $S \subset Y$, *where* $\mathbf{K}$ *and* $\mathbf{F}$ *are infinite non-discrete non-Archimedean fields with multiplicative ultra-norms* $|*|_{\mathbf{K}}$ *and* $|*|_{\mathbf{F}}$. *Then from* $J_\mu = Y$ *it follows that* $\mu = 0$.

Proof. Since μ is σ-finite, then there are $(Y_j : j \in H) \subset Bco(Y)$ such that $Y = \bigcup_{j\in H} Y_j$ and $0 < \|\mu|Bco(Y_j)\| \leq 1$ for each j, where $H \subset \mathbf{N}$, $Y_j \cap Y_l = \emptyset$ for each $j \neq l$. If $card(H) =$

$\aleph_0$, then we define a function $f(x) = s^j/\|Y_j\|_\mu$ for μ with values in $\mathbf{F}$, where s is fixed with $0 < |s|_F < 1$, $s \in \mathbf{N}$). Then we define a measure $\nu(A) = \int_A f(x)\mu(dx)$, $A \in Bco(Y)$. Therefore, $\|Y\|_\nu \le 1$ and $J_\nu = Y$, since $f \in L(Y,\mu,\mathbf{F})$. Hence it is sufficient to consider μ with $\|\mu\| \le 1$ and $\mu(Y) = 1$. For each $n \in \mathbf{N}$ in view of the Radonian property of Y there exists a compact $X_n \subset Y$ such that $\|Y \setminus X_n\|_\mu < s^{-n}$. In Y there is a countable everywhere dense subset $(x_j : j \in \mathbf{N})$, hence $Y = \bigcup_{j\in\mathbf{N}} B(Y,x_j,r_l)$ for each $r_l > 0$, where $B(Y,x,r_l) = [y \in Y : d(x,y) \le r_l]$, d is an ultra-metric in Y, i.e. $d(x,z) \le max(d(x,y),\, d(y,z))$, $d(x,z) = d(z,x)$, $d(x,x) = 0$, $d(x,y) > 0$ for $x \ne y$ for each $x,y,z \in Y$. Therefore, for each $r_l = 1/l$, $l \in \mathbf{N}$ there exists $k(l) \in \mathbf{N}$ such that $\|Y \setminus X_{n,l}\|_\mu < s^{-n-l}$ due to compactness of $Y_c = [y \in Y : N_\mu(y) \ge c]$ for each $c > 0$, where $X_{n,l} := \bigcup_{j=1}^{k(l)} B(Y,x_j,r_l)$, consequently, $\|Y \setminus X_n\|_\mu \le s^{-n}$ for $X_n := \bigcap_{l=1}^{\infty} X_{n,l}$. The subsets X_n are compact, since X_n are closed in Y and the metric d on X_n is completely bounded and Y is complete (see also Theorems 3.1.2 and 4.3.29 [Eng86]). Then $0 < \|X\|_\mu \le 1$ for $\|Y \setminus X\|_\mu = 0$ and for $X := span_{\mathbf{K}}(\bigcup_{n=1}^{\infty} X_n)$.

The sets $\tilde{Y}_n := co(Y_n)$ are nowhere dense in Y for $Y_n = \bigcup_{l=1}^{n} X_l$, consequently, $span_{\mathbf{K}} Y_n$ are nowhere dense in Y. Moreover, $(Y \setminus \bigcup_{n=1}^{\infty} Y_n) \ne \emptyset$ is dense in Y due to the Baire category theorem (see also § § 3.9.3 and 4.3.26 [Eng86]). Therefore, $y + X \subset Y \setminus X$ for $y \in Y \setminus X$ and from $J_\mu = Y$ it follows that $\|X\|_\mu = 0$, since $\|y+X\|_\mu = 0$ (see § § 2.32 and 3.12 above). Hence we get the contradiction, consequently, $\mu = 0$.

3.14. Corollary. *If Y is a Banach space or a complete countably-ultra-normable infinite-dimensional over $\mathbf{K}$ space, $\mu : Bco(Y) \to \mathbf{K_s}$, $\mathbf{K}$ and $\mathbf{F}$ are the same as in* § 3.13 *and $J_\mu = Y$, then $\mu = 0$.*

Proof. The space Y is evidently complete and ultra-metrizable, since its topology is given by a countable family of ultra-norms.

Recall the theorem about hulls of compact sets (see also § (5.7.5) in [NB85]).

(NBi). In any topological vector space X the balanced hull H_b of a bounded or compact set H is again totally bounded or compact respectively.

$(NBii)$. If H is a totally bounded subset of a locally convex space X, then so is its convex hull H_c and therefore its disked hull H_{bc}.

$(NBiii)$. If H is a compact subset of a locally convex space X, then its convex hull H_c and disked hull H_{bc} are compact if and only if H_c and H_{bc} are complete respectively. If X is complete, then $cl_X H_c$ and $cl_X H_{bc}$ are compact.

Due to this theorem $co(S)$ is nowhere dense in Y for each compact S in Y, since $co(S) = cl(S_{bc})$ is compact in Y and does not contain in itself any open subset in Y, since Y is infinite dimensional over the field $\mathbf{K}$.

3.15. Theorem. *Let X be a separable Banach space over a locally compact infinite field $\mathbf{K}$ with a nontrivial normalization such that either $\mathbf{K} \supset \mathbf{Q_p}$ or $char(\mathbf{K}) = p > 0$. Then there are probability measures μ on X with values in $\mathbf{K_s}$ $(s \ne p)$ such that μ are quasi-invariant relative to a dense $\mathbf{K}$-linear subspace J_μ.*

Proof. Let $S(j,n) := p^j B(\mathbf{K},0,1) \setminus p^{j+1} B(\mathbf{K},0,1)$ for $j \in \mathbf{Z}$ and $j \le n$, $S(n,n) := p^n B(\mathbf{K},0,1)$, w' be the Haar measure on $\mathbf{K}$ considered as the additive group (see also [Bou63-69, HR79, Roo78]) with values in $\mathbf{K_s}$ for $s \ne p$. Then for each $c > 0$ and $n \in \mathbf{N}$ there are measures m on $Bf(\mathbf{K})$ such that $m(dx) = f(x)v(dx)$, $|\ f(x)\ | > 0$ for each $x \in \mathbf{K}$ and $|\ m(p^n B(\mathbf{K},0,1)) - 1\ | < c$, $m(\mathbf{K}) = 1$, $|\ m\ |\ (E) \le 1$ for each $E \in Bco(\mathbf{K})$, where $v = w'$, $v(B(\mathbf{K},0,1)) = 1$. Moreover, we can choose f such that a density $m_a(dx)/m(dx) =: d(m;a,x)$ be continuous by $(a,x) \in \mathbf{K^2}$ and for each $c' > 0$, x and

$|a| \le p^{-n} : |d(m;a,x) - 1| < c'$. Let $f|_{S(j,n)} := a(j,n)$ be locally constant, for example, $a(j,n) = (1-s)(1-1/p)s^{2n-1-j}p^{-n}$ for $j < n$, $a(n,n) = (1-s^{-n})p^{-n}$. Then taking $f+h$ and using $h(x)$ with $0 < \sup_x |h(x)/f(x)| = c" \le 1/s^n$ we get $|y_a(dx)/y(dx)| = |m_a(dx)/m(dx)|$, where $y(dx) = (f+h)(x)m(dx)$. More generally it is possible to take $g \in L(\mathbf{K}, Bco(\mathbf{K}), w', \mathbf{K_s})$ such that $g(x) \neq 0$ for w'-almost every $x \in \mathbf{K}$ and $\|g\| = 1$ and $\prod_{n=1}^{\infty} \beta_n > 0$ converges for each $y = \{y_n : y_n \in \mathbf{K}, n \in \mathbf{N}\}$ in a proper dense subspace J in $X = c_0$, where $g_n(x) := g(x/a_n)$, $\lim_{n\to\infty} a_n = 0$, $0 \neq a_n \in \mathbf{K}$ for each $n \in \mathbf{N}$, $\beta_n := \|\rho_n\|_{\phi_n}$, $\rho_n(x) := \mu_n(dx)/\nu_n(dx)$, $\phi_n(x) := N_{\lambda_n}(x)$, $\lambda_n(dx) := g_n(x)w'(dx/a_n)$, then use Theorem 3.5 for the measure $\nu_n(dx) := g_n(x)w'(dx/a_n)$ and its shifted measure $\mu_n(dx) := \nu_n(-y_n + dx)$.

Let $\{m(j;dx)\}$ be a family of measures on $\mathbf{K}$ with the corresponding sequence $\{k(j)\}$ such that $k(j) \le k(j+1)$ for each j and $\lim_{i\to\infty} k(i) = \infty$, where $m(j;dx)$ corresponds to the partition $[S(i,k(j))]$. The Banach space X is isomorphic with $c_0(\omega_0, \mathbf{K})$, since $\mathbf{K}$ is spherically complete. It has the orthonormal basis $\{e_j : j = 1,2,\dots\}$ and the projectors $P_j x = (x(1),\dots,x(j))$ onto $\mathbf{K^j}$, where $x = x(1)e_1 + x(2)e_2 + \cdots$. Then there exists a cylindrical measure μ generated by a consistent family of measures $y(j,B) = b(j,E)$ for $B = P_j^{-1}E$ and $E \in Bf(\mathbf{K}^j)$ [Bou63-69, DF91] where $b(j,dz) = \otimes[m(j;dz(i)) : i = 1,\dots,j], z = (z(1),\dots,z(j))$. Let $L := L(t,t(1),\dots,t(l);l) := \{x : x \in X$ and $|x(i)| \le p^a, a = -t - t(i)$ for $i = 1,\dots,l$, and $a = -k(j)$ for $j > l\}$, then L is compact in X, since X is Lindelöf and L is sequentially compact (see also [Eng86]). Therefore, for each $c > 0$ there exists L such that $\|X \setminus L\|_\mu < c$ due to the choice of $a(j,n)$.

In view of the non-Archimedean analog of the Prohorov theorem for measures with values in $\mathbf{K_s}$ reminded above (see also 7.6(ii)[Roo78]) and due to Lemma 2.3 μ has the countably-additive extension on $Bf(X)$, consequently, also on the complete σ-field $Af(X,\mu)$ and μ is the Radon measure.

Let $z' \in span_{\mathbf{K}}\{e_j : j = 1,2,\dots\}$ and $z" = \{z(j) : z(j) = 0$ for $j \le l$ and $z(j) \in S(n,n), j = 1,2,\dots,n = k(j)\}$, $l \in \mathbf{N}$, $z = z' + z"$. Now take the restriction of μ on $Bco(X)$. In view of Theorems 2.19, 3.5 above and also the lemma about isometric mappings I.1.4 [Roo78] recalled above (see also § I.3.20 above) there are $m(j;dz(j)$ such that $\rho_\mu(z,x) = \prod\{d(j;z(j),x(j)) : j = 1,2,\dots\} = \mu_z(dx)/\mu(dx) \in L((X,\mu,Bco(X)),\mathbf{K_s})$ for each such z and $x \in X$, where $d(j;*,*) = d(m(j;*),*,*)$ and $\mu_z(X) = \mu(X) = 1$.

3.15.1. Theorem. *Let X be a Banach space of separable type over a locally compact non-Archimedean field $\mathbf{K}$ and J be a dense proper $\mathbf{K}$-linear subspace in X such that the embedding operator $T : J \hookrightarrow X$ is compact and nondegenerate, $ker(T) = \{0\}$. Then a set $\mathcal{M}(X,J)$ of probability $\mathbf{K_s}$-valued measures μ on $Bco(X)$ quasi-invariant relative to J is of cardinality $card(\mathbf{K_s})^{\mathsf{c}}$. If J', $J' \subset J$, is also a dense $\mathbf{K}$-linear subspace in X, then $\mathcal{M}(X,J') \supset \mathcal{M}(X,J)$.*

Proof. As in the proof of Theorem 3.20.1 choose for a given compact operator T an orthonormal base in X in which T is diagional and X is isomorphic with c_0 over $\mathbf{K}$ such that in its standard base $\{e_j : j \in \mathbf{N}\}$ the operator T is such that $Te_j = a_j e_j$, $0 \neq a_j \in \mathbf{K}$ for each $j \in \mathbf{N}$, $\lim_{j\to\infty} a_j = 0$ (see Appendix). As in Theorem 3.15 take $g_n \in L(\mathbf{K}, Bco(\mathbf{K}), w'(dx/a_n), \mathbf{K_s})$, $g_n(x) \neq 0$ for v-a.e. $x \in \mathbf{K}$ and $\|g_n\| = 1$ for each n, for which converges $\prod_{n=1}^{\infty} \beta_n > 0$ for each $y \in J$ and such that $\prod_{n=1}^{m} g_n(x_n)w'(dx_n/a_n) =: \nu_{L_n}(dx^n)$ satisfies conditions of Lemma 2.3, where $\beta_n := \|\rho_n\|_{\phi_n}$, $0 \neq a_n \in \mathbf{K}$ for each $n \in \mathbf{N}$, $\rho_n(x) := \mu_n(dx)/\nu_n(dx)$, $\phi_n(x) := N_{\lambda_n}(x)$, $\lambda_n(dx) := g_n(x)w'(dx/a_n)$, then use Theorem 3.5 for the measure $\nu_n(dx) := g_n(x)w'(dx/a_n)$ and $\mu_n(dx) := \nu_n(-y_n + dx)$, $x^n := (x_1,\dots,x_n)$,

$x_1,\dots,x_n \in \mathbf{K}$ for each $n \in \mathbf{N}$. The family of such sequences of functions $\{g_n : n \in \mathbf{N}\}$ has the cardinality $card(\mathbf{K_s})^{\mathsf{c}}$, since in $L(\nu)$ the subspace of step functions is dense and $card(Bco(X)) = \mathsf{c}$. The family of all $\{g_n : n\}$ satisfying conditions above for J also satisfies such conditions for J'. From which the latter statement of this theorem follows.

3.16. Note. For a given $m = w'$ (see above) new suitable measures may be constructed, if to use images of measures $m^g(E) = m(g^{-1}(E))$ such that for a diffeomorphism $g \in Diff^1(\mathbf{K})$ (see § A.3) we have $m^{g^{-1}}(dx)/m(dx) = |(g'(g^{-1}(x))|_{\mathbf{K}}$, where $|*|_{\mathbf{K}} = mod_{\mathbf{K}}(*)$ is the modular function of the field $\mathbf{K}$ associated with the Haar measure on $\mathbf{K}$, at the same time $|*|_{\mathbf{K}}$ is the multiplicative norm in $\mathbf{K}$ consistent with its uniformity [Wei73]. Indeed, for $\mathbf{K}$ and $X = \mathbf{K^j}$ with $j \in \mathbf{N}$ and the Haar measure $v = w'$ on X, $v_X := v$ with values in $\mathbf{K_s}$ for $s \neq p$ and for a function $f \in L(X, v, \mathbf{K_s})$ we have: $\int_{g(A)} f(x)v(dx) = \int_A f(g(y))|g'(y)|_{\mathbf{K}} v(dy)$, where $mod_{\mathbf{K}}(\lambda)v(dx) := v(\lambda dx)$, $\lambda \in \mathbf{K}$, since $v(B(X,0,p^n)) \in \mathbf{Q}$, $N_v(x) = 1$ for each $x \in X$, consequently, from $f_k \to f$ in $L(g(A), v, \mathbf{K_s})$ whilst $k \to \infty$ it follows that $f_k(g(x)) \to f(g(x))$ in $L(A, v, \mathbf{K_s})$, where f_k are locally constant, A is compact and open in X.

Henceforward, quasi-invariant measure μ on $Bco(c_0(\omega_0, \mathbf{K}))$ constructed with the help of projective limits or sequences of weak distributions of probability measures $(\mu_{H(n)} : n)$ are considered, for example, as in Theorem 3.15 such that

(i) $\mu_{H(n)}(dx) = f_{H(n)}(x)v_{H(n)}(dx)$, $dim_{\mathbf{K}} H(n) = m(n) < \aleph_0$ for each $n \in \mathbf{N}$, where $f_{H(n)} \in L(H(n), v_{H(n)}, \mathbf{K_s})$, $H(n) \subset H(n+1) \subset \cdots$, $cl(\bigcup_n H(n) = c_0(\omega_0, \mathbf{K})$, if it is not specified in another manner.

In accordance with the Schikhof's Theorem 8.9 [Roo78]: if G is a zero-dimensional topological group with unit element e, ν is a tight measure, $\nu \in M(G)$, $\nu \neq 0$, then the following conditions are equivalent:

(α) the mapping $s \mapsto \nu_s$ is continuous $G \to M(G)$;

(β) this mapping $s \mapsto \nu_s$ is continuous at e;

(γ) G has a left Haar measure h and $\nu \in L(G)$, where $L(G)$ denotes the set of tight absolutely continuous measures relative to the Haar measure h on G.

Each $\nu \in L(G)$ can be written in the form $\nu = f\mathsf{h}$, where $f \in C^0_\infty(G)$ is a continuous function so that for each $b > 0$ there exists a compact subset E_b in G with $|f| < b$ outside a compact subset E_b.

For probability quasi-invariant measure with values in $\mathbf{K_s}$, if shifts $x \mapsto x+y$ by $y \in H(n)$ are continuous from $H(n)$ to $\mathsf{M}(H(n))$ (see § 2.1), that is, $y \to \mu^y_{H(n)}$, where $\mu_{H(n)}(y+A) =: \mu^y_{H(n)}(A)$ for $A \in Bco(H(n))$, then due to the Schikhof's theorem about tight measures on zero-dimensional groups $\mu_{H(n)}$ satisfies (i).

As will be seen below such measures μ are quasi-invariant relative to families of the cardinality $\mathsf{c} = card(\mathbf{R})$ of linear and non-linear transformations $U : X \to X$. Moreover, for each V open in X we have $\|V\|_\mu > 0$, when $f_{H(n)}(x) \neq 0$ for each $n \in \mathbf{N}$ and $x \in H(n)$.

Let μ be a probability quasi-invariant measure satisfying (i) and $(e_j : j)$ be orthonormal basis in M_μ, $H(n) := span_{\mathbf{K}}(e_1,\dots,e_n)$, we denote by

$\hat{\rho}_\mu(a,x) = \hat{\rho}(a,x) = \lim_{n\to\infty} \rho^n(P_n a, P_n x)$,

$\rho^n(P_n a, P_n x) := f_{H(n)}(P_n(x-a))/f_{H(n)}(P_n x)$ for each a and x for which this limit exists and $\hat{\rho}(a,x) = 0$ in the contrary case, where $P_n : X \to H(n)$ are chosen consistent projectors. Let $\rho(a,x) = \hat{\rho}(a,x)$, if $\mu_a(X) = \mu(X)$ and $\hat{\rho}(a,x) \in L(X, \mu, \mathbf{K_s})$ as a function by x and $\|X\|_{N_\nu} = 1$, where $\nu(dx) := \hat{\rho}(a,x)\mu(dx)$, $\rho(a,x)$ is not defined when $\mu_a(X) = \mu(X)$ or

$\|X\|_{N_\nu} \neq 1$, this condition of the equality to 1 may be satisfied, for example, for continuous $f_{H(n)}$ with continuous $\hat{\rho}(a,x) \in L(\mu)$ by x for each given a, if $\lim_n \rho^n(a,x)$ converges uniformly by x. If for some another basis $(\tilde{e}_j : j)$ and $\tilde{\rho}$ is accomplished

(ii) $\|X \setminus S\|_\mu = 0$, then $\rho(a,x)$ is called regularly dependent from a basis, where $S := \bigcap_{a\in M_\mu}[x : \rho(a,x) = \tilde{\rho}(a,x)])$.

3.17. Lemma. *Let μ be a probability measure, $\mu : Bco(X) \to \mathbf{K_s}$, X be a Banach space over $\mathbf{K}$, suppose that for each basis $(\tilde{e}_j : j)$ in M_μ a quasi-invariance factor $\tilde{\rho}$ satisfies the following conditions:*

(1) *if $\tilde{\rho}(a_j,x)$, $j = 1,\dots,N$, are defined for a given $x \in X$ and for each $\lambda_j \in \mathbf{K}$ then a function $\tilde{\rho}(\sum_{j=1}^n \lambda_j a_j, x)$ is continuous by λ_j, $j = 1,\dots,N$;*

(2) *there exists an increasing sequence of subspaces $H(n) \subset M_\mu$, $cl(\bigcup_n H(n)) = X$, with projectors $P_n : X \to H(n)$, $B \in Bf(X)$, $\|B\|_\mu = 0$ such that $\lim_{n\to\infty} \tilde{\rho}(P_n a,x) = \tilde{\rho}(a,x)$ for each $a \in M_\mu$ and $x \notin B$ for which is defined $\rho(a,x)$. Then $\rho(a,x)$ depends regularly from the basis.*

Proof. There exists a subset S dense in each $H(n)$, hence $\|B'\|_\mu = 0$ for $B' = \bigcup_{a\in S}[x : \rho(a,x) \neq \tilde{\rho}(a,x)]$. From (1) it follows that $\tilde{\rho}(a,x) = \rho(a,x)$ on each $H(n)$ for $x \notin B'$. From $span_{\mathbf{K}} S \supset H(n)$ and (2) it follows that $\rho(a,x) = \tilde{\rho}(a,x)$ for each $a \in M_\mu$ and $x \in X \setminus (B' \cup B)$, consequently, Condition 3.16(ii) is satisfied, since from $\rho(a,x) \in L(\mu)$ it follows that $\tilde{\rho}(a,x) \in L(\mu)$ as the function by x.

3.18. Lemma. *If a probability quasi-invariant measure $\mu : Bco(X) \to \mathbf{K_s}$ satisfies Condition* 3.16(i), *then there exists a compact operator $T : X \to X$ such that $M_\mu \subset (TX)^\sim$, where X is the Banach space over* $\mathbf{K}$.

Proof. Products of tight measures are tight measures (see also Theorem 7.28 [Roo78]), whence for $\mu_{H(n)}(dx) = \bigotimes_{j=1}^{m(n)} \mu_{\mathbf{K}e(j)}(dx_j)$ is accomplished $N_{\mu_{H(n)}}(x) = \prod_{j=1}^{m(n)} N_{\mu_{\mathbf{K}e(j)}}(x_j)$, where $x = (x_1,..,x_{m(n)})$, $x_j \in \mathbf{K}$. From Theorem 7.6[Roo78] formulated above and Lemma I.2.5 it follows that for each $1 > c > 0$ there are $R_j = R_j(c)$ with $[x_j : N_{\mu_{\mathbf{K}e(j)}}(x_j) \geq c] \subset B(\mathbf{K},0,R_j)$ and $\lim_{j\to\infty} R_j = 0$.

Choosing $c = c(n) = s^{-n}$, $n \in \mathbf{N}$ and using $\prod_{j=1}^\infty = \varepsilon_j = 0$ whilst $0 < \varepsilon_j < c < 1$ for each j we get that there exists a sequence $[r_j : j]$ for which $card[j : |a_j| > r_j] < \aleph_0$ for each $a \in M_\mu$, since $[x \in X : |x_j| \leq r_j$ for all $j]$ is a compact subgroup in X, where $a = (a_j : j)$, $a_j \in \mathbf{K}$, $r_j > 0$, $\lim_j r_j = 0$. Therefore, $M_\mu \subset (TX)^\sim$ for $T = diag(T_j : j)$ and $|T_j| \geq r_j$ for $j \in \mathbf{N}$.

3.19. Let X be a Banach space over $\mathbf{K}$, $|*|_{\mathbf{K}} = mod_{\mathbf{K}}(*)$, $U : X \to X$ be an invertible linear operator, $\mu : Bco(X) \to \mathbf{K_s}$ be a probability quasi-invariant measure.

The uniform convergence of a (transfinite) sequence of functions on $Af(V,\nu)$-compact subsets of a topological space V is called the Egorov condition, where ν is a measure on V.

Theorem. *Let pairs $(x-Ux,x)$ and $(x-U^{-1},x)$ be in $dom(\tilde{\rho}(a,x))$, where $dom(f)$ denotes a domain of a function f, $\tilde{\rho}(x-Ux,x) \neq 0$, $\tilde{\rho}(x-U^{-1}x,x) \neq 0$ $(mod\ \mu)$ and μ satisfies Condition 3.16(i), also $\tilde{\rho}(\tilde{P}_n(x-Ux),x) =: \tilde{\rho}_n(x) \in L(\mu)$ and $\tilde{\rho}(\tilde{P}_n(x-U^{-1}x,x) =: \bar{\rho}_n(x) \in L(\mu)$ converge uniformly on $Af(X,\mu)$-compact subsets in X such that there exists $g \in L(\mu)$ with $|\tilde{\rho}_n(x)| \leq |g(x)|$ and $|\bar{\rho}_n(x)| \leq |g(x)|$ for each $x \in X$ and each projectors $\tilde{P}_n : X \to \tilde{H}(n)$ with $cl(\bigcup_n \tilde{H}(n)) = X$, $\tilde{H}(n) \subset \tilde{H}(n+1) \subset \dots$, that is, Egorov conditions are satisfied for $\tilde{\rho}_n$ and $\bar{\rho}_n$. Then $\nu \sim \mu$ and*

$$(i)\ \nu(dx)/\mu(dx) = |det(U)|_{\mathbf{K}}\tilde{\rho}(x-U^{-1}x,x),$$

if ρ *depends regularly from the base, then* $\tilde{\rho}$ *may be substituted by* ρ *in formula* (i), *where* $\nu(A) := \mu(U^{-1}A)$ *for each* $A \in Bco(X)$.

Proof. In view of Lemma 3.18 there exists a compact operator $T : X \to X$ such that $M_\mu \subset (TX)^\sim$, consequently, $(U - I)$ is a compact operator, where I is the identity operator. From the invertibility of U it follows that $(U^{-1} - I)$ is also compact, moreover, there exists $det(U) \in \mathbf{K}$. Let g be a continuous bounded function, $g : \tilde{H}(n) \to \mathbf{K_s}$, whence

$$\int_X \phi(x)\nu(dx) = \int_{\tilde{H}(n)} g(x)[f_{\tilde{H}(n)}(U^{-1}x)/f_{\tilde{H}(n)}(x)]|det(U_n)|_{\mathbf{K}}\mu_{\tilde{H}(n)}(dx),$$

for $\phi(x) = g(\tilde{P}_n x)$, where subspaces exist such that $\tilde{H}(n) \subset X$, $(U^{-1} - I)\tilde{H}(n) \subset \tilde{H}(n)$, $cl(\bigcup_n \tilde{H}(n)) = X$, $U_n := \hat{r}_n(U)$, $r_n = \tilde{P}_n : X \to \tilde{H}(n)$ (see § § I.3.8 and II.3.16), $\tilde{H}(n) \subset \tilde{H}(n+1) \subset \cdots$ due to compactness of $(U - I)$.

In view of the non-Archimedean analog of the Lebesgue convergence theorem due to fulfillment of the Egorov conditions for $\tilde{\rho}_n$ and $\overline{\rho}_n$ (see also §7.6 [MS63] or § 7.F [Roo78]) $J_m = J_{m,\rho}$, since $\tilde{\rho}(x - U^{-1}x, x) \in L(\mu)$, where

$$J_m := \int_X g(\tilde{P}_m x)\nu(dx) \text{ and}$$

$$J_{m,\rho} := \int_X g(\tilde{P}_m x)\tilde{\rho}(x - U^{-1}x, x)|det(U)|_{\mathbf{K}}\mu(dx).$$

Indeed, there exists n_0 such that $|u(i,j) - \delta_{i,j}| \le 1/p$ for each i and $j > n_0$, consequently, $|det(U_n)|_{\mathbf{K}} = |det(U)|_{\mathbf{K}}$ for each $n > n_0$. Then due to Condition 3.16.(i) and the Egorov conditions (see also § 3.3) there exists

$$\lim_{n\to\infty}[\mu_{\tilde{H}(n)}(d\tilde{P}_n x)/\nu_{\tilde{H}(n)}(d\tilde{P}_n x)] = \mu(dx)/\nu(dx) \quad (mod\ \nu).$$

Further analogously to the proof of Theorem I.3.24 above.

3.20. Let X be a Banach space over $\mathbf{K}$, $|*|_{\mathbf{K}} = mod_{\mathbf{K}}(*)$ with a probability quasi-invariant measure $\mu : Bco(X) \to \mathbf{K_s}$ and Condition 3.16(i) be satisfied, also let U fulfils the following conditions:

$$(i)\ U(x) \text{ and } U^{-1}(x) \in C^1(X,X) \text{ (see also § A.3);}$$

$$(ii)\ (U'(x) - I) \text{ is compact for each } x \in X;$$

$$(iii)\ (x - U^{-1}(x)) \text{ and } (x - U(x)) \in J_\mu \text{ for } \mu-\text{a.e. } x \in X;$$

$$(iv) \text{ for } \mu\text{-a.e. } x \text{ pairs } (x - U(x);x) \text{ and } (x - U^{-1}(x);x)$$

are contained in a domain of $\rho(z,x)$ such that $\rho(x - U^{-1}(x), x) \neq 0$, $\rho(x - U(x), x) \neq 0$ $(mod\ \mu)$;

$$(v)\ \|X \setminus S'\|_\mu = 0,$$

where $S' := ([x :\ \rho(z,x) \text{ is defined and continuous by } z \in L])$ for each finite-dimensional $L \subset J_\mu$;

$$(vi) \text{ there exists } S \text{ with } \|S\|_\mu = 0 \text{ and for each}$$

$x \in X \setminus S$ and for each z for which there exists $\rho(z,x)$ satisfying the following condition: $\lim_{n\to\infty} \rho(P_n z,x) = \rho(z,x)$ and the convergence is uniform for each finite-dimensional $L \subset J_\mu$ by z in $L \cap [x \in J_\mu : |x| \le c]$, where $c > 0$, $P_n : X \to H(n)$ are projectors onto finite-dimensional subspaces $H(n)$ over $\mathbf{K}$ such that $H(n) \subset H(n+1)$ for each $n \in \mathbf{N}$ and $cl \cup \{H(n) : n\} = X$;

(vii) there exists n for which for all $j > n$ and $x \in X$ mappings

$V(j,x) := x + P_j(U^{-1}(x) - x)$ and $U(j,x) := x + P_j(U(x) - x)$ are invertible and $\lim_j | \det U'(j,x) | = | \det U'(x) |$, $\lim_j | \det V'(j,x) | = 1/ | \det U'(x) |$ with the Egorov convergence in (vi) by z for $\rho(P_n z,x)$ and in (vii) by x for $|det(U'(j,x))|$ and $|det(V'(j,x))|$ for μ with values in $\mathbf{K_s}$.

Theorem. *The measure* $\nu(A) := \mu(U^{-1}(A))$ *is equivalent to* μ *and*

$$(i)\ \nu(dx)/\mu(dx) = | \det U'(U^{-1}(x)) |_{\mathbf{K}}\, \rho(x - U^{-1}(x), x).$$

Proof. The beginning of the proof is analogous to that of § I.3.25. Due to Conditions (vi, vii) we get $\lim_n \rho(x - V_n^{-1}x, x) = \rho(x - U_1^{-1}x, x)$ in $L(\mu)$ due to the Egorov conditions. Then $J_1 = J_{1,\rho}$ due to the Lebesgue convergence theorem, where

$$J_1 = \int_X f(U_1 x)\mu(dx),$$

$$J_{1,\rho} := \int_X f(x)\rho(x - U_1^{-1}x, x)|detU_1|_{\mathbf{K}}\mu(dx)$$

for continuous bounded function $f : X \to \mathbf{K_s}$. Analogously for U_1^{-1} instead of U_1. Using instead of f the function $\bar{\Phi}(U_1^{-1}x) := f(x)\rho_\mu(x - U_1^{-1}x, x)$ and Properties 3.10 we get that $\rho_\mu(U_1 x - x, U_1 x)\rho_\mu(x - U_1 x, x) = 1$ $(mod\ \mu)$. Therefore, for $U = U_1 U_2$ with diagonal U_1 and upper triangular U_2 and lower triangular U_3 operators with finite-dimensional over $\mathbf{K}$ subspaces $(U_j - I)X$, $j = 1,2,3$, the following equation is accomplished $\int_X f(Ux)\mu(dx) = \int_X f(x)\rho_\mu(x - U^{-1}x, x)|detU|_{\mathbf{K}}\mu(dx)$. If either $(S^{-1}U - I)X = L$ or $(U^{-1}S - I)X = L$, then from the decomposition given in (I) $U = SU_2U_1U_3$, we have either $(U_j - I)X = L$ or $(U_j^{-1} - I)X = L$ respectively, $j = 1,2,3$ due to formulas from § A.1, since corresponding non-major minors are equal to zero.

If U is an arbitrary linear operator satisfying the conditions of this theorem, then from $(iv - vi)$ and (I, II) for each continuous bounded function $f : X \to \mathbf{K_s}$ we have $J = J_\rho$, where

$$J := \int_X f(U(x))\mu(dx) \text{ and}$$

$$J_\rho := \int_X f(x)\rho_\mu(x - U^{-1}(x), x)|detU|_{\mathbf{K}}\mu(dx).$$

Analogously for U^{-1}, moreover, $\rho(x - U^{-1}(x), x)|detU|_{\mathbf{K}} =: h(x) \in L(\mu)$, $h(x) \ne 0$ $(mod\ \mu)$, since there exists $detU$.

Suppose U is polygonal (see § I.3.25). Then U^{-1} is also polygonal, $U'(x) = V(j)$ for $x \in Y(j)$ and $\int_X f(a(i) + V(i)x)\mu(dx) = \int f(a(i) + x)\rho_\mu(x - V^{-1}(i)x, x) \times |det(V(i))|_{\mathbf{K}}\mu(dx)$ for each continuous bounded $f : X \to \mathbf{K_s}$ and each i. From $a(j) \in M_\mu$ and § 3.10 we get

$\int_X f(a(j)+V(j)x)\mu(dx) = \int_X f(x)\rho(x-V(j)^{-1}(x-a(j)),x)|detV(j)|_{\mathbf{K}}\mu(dx)$. Let $H_{k,j} := [x \in X : V(k)^{-1}x = V(j)^{-1}x]$, assume without loss of generality that $V(k) \neq V(j)$ or $a(k) \neq a(j)$ for each $k \neq j$, since $Y(k) \neq Y(j)$ (otherwise they may be united). Therefore, $H_{k,j} \neq X$. If $\|H_{k,j}\|_\mu > 0$, then from $X \ominus H_{k,j} \supset \mathbf{K}$ it follows that $M_\mu \subset H_{k,j}$, but $cl(H_{k,j}) = H_{k,j}$ and $cl(M_\mu) = X$. This contradiction means that $\|A\|_\mu = 0$, where $A = [x : V(k)^{-1}(x-a(k)) = V(j)^{-1}(x-a(j))]$. Then $\int_X f(U(x))\mu(dx) = \int_X f(x)\rho(x-U^{-1}(x),x)\ |detU'(x)|_{\mathbf{K}}^{-1}\mu(dx)$.

Then as in § I.3.25(V) for the construction of the sequence $\{U(j,*) : j\}$ it is sufficient to construct a sequence of polygonal functions $\{a(i,j;x)\}$, that is $a(i,j;x) = l_k(i,j)(x)+a_k$ for $x \in Y(k)$, where $l_k(i,j)$ are linear functionals, $a_k \in \mathbf{K}$, $Y(k)$ are closed in X, $Int(Y(j)) \cap Int(Y(k)) = \emptyset$ for each $k \neq j$, $\bigcup_{k=1}^m Y(k) = X$, $m \leq \aleph_0$. For each $c > 0$ there exists $V_c \subset X$ with $\|X \setminus V_c\| < c$, the functions $s(i,j;x)$ and $(\bar{\Phi}^1 s(i,j;*))(x,e(k),t)$ are equiuniformly continuous (by $x \in V_c$ and by $i,j,k \in \mathbf{N}$) on V_c. Choosing $c = c(n) = s^{-n}$ and using δ-nets in V_c we get a sequence of polygonal mappings $(W_n : n)$ converging by its matrix elements by Egorov in the Banach space $L(X,\mu,\mathbf{K_s})$, from Condition (i) it follows that it may be chosen equicontinuous for matrix elements $s(i,j;x)$, $ds(i,j;x)$ and $s(i,P_jx)$ by i,j (the same is true for U^{-1}).

Then calculating integrals as above for W_n with functions f, using the Lebesgue convergence theorem we get the equalities analogous to written in § I.3.25(III) for J and J_ρ of the general form. From $\nu(dx)/\mu(dx) \neq 0$ $(mod\ \mu)$ and § 2.19 we get the statement of the theorem.

3.21. Examples. Let X be a Banach space over the field $\mathbf{K}$ with the normalization group $\Gamma_{\mathbf{K}} = \Gamma_{\mathbf{Q_p}}$. We consider a diagonal compact operator $T = diag(t_j : j \in \mathbf{N})$ in a fixed orthonormal basis $(e_j : j)$ in X such that $kerT := T^{-1}0 = \{0\}$. Let $\nu'_j(dx_j) = C'(\xi_j)s^{-q\min(0,ord_p((x_j-x_j^0)/\xi_j))}\nu(dx_j)$ for the Haar measure $\nu : Bco(\mathbf{K}) \to \mathbf{Q_s}$, then $\nu'_j(Bf(\mathbf{K})) \subset \mathbf{C_s}$. We choose constant functions $C'(\xi_j)$ such that ν'_j be a probability measure, where $x^0 = (x_j^0 : j) \in X$, $x = (x_j : j) \in X$, $x_j \in \mathbf{K}$.

With the help of products $\bigotimes_j \nu'_j(dx_j)$ as in § 3.15 we can construct a probability quasi-invariant measure μ^T on X with values in $\mathbf{C_s}$, since $cl(TX)$ is compact in X and $span_{\mathbf{K}}(e_j : j) =: H \subset J_\mu$. From $\bigcap_{\lambda \in B(\mathbf{K},0,1)\setminus 0} cl(\lambda TX) = \{0\}$ we may infer that for each $c > 0$ there exists a compact $V_c(\lambda) \subset X$ such that $\|X \setminus V_c(\lambda)\|_\mu < c$ and $\bigcap_{\lambda \neq 0} V_c(\lambda) = \{0\}$, consequently,

$$\lim_{|\lambda| \to 0} \int_X f(x)\mu^{\lambda T}(dx) = f(0) = \delta_0(f),$$

hence $\mu^{\lambda T}$ is weakly converging to δ_0 whilst $|\lambda| \to 0$ for the space of bounded continuous functions $f : X \to \mathbf{C_s}$.

From Theorem 3.6 we conclude that from $\sum_{j=1}^\infty |y_j/\xi_j|_p^q < \infty$ it follows $y \in J_{\mu^T}$. Then for a linear transformation $U : X \to X$ from $\sum_j |\tilde{e}_j(x-U(x))/\xi_j|_p^q < \infty$ it follows that $x - U(x) \in J_\mu$ and a pair $(x-U(x),x) \in dom(\rho(a,z))$. Moreover, for ρ corresponding to μ^T conditions (v) and (vi) in § 3.20 are satisfied. Therefore, for such y and $S \in Af(X,\mu)$ a quantity $|\mu(ty+S)-\mu(S)|$ is of order of smallness $|t|^q$ whilst $t \to 0$, hence they are pseudo-differentiable of order b for $0 < Re(b) < q$ (see also § 4 below).

It is interesting also to discuss a way of solution of one problem formulated in [KE92] that there does not exist a σ-additive $\mathbf{Q_p}$-valued measure with values in X over $\mathbf{Q_p}$ such that it would be an analog of the classical Gaussian measure. In the classical case this means in particular a quasi-invariance of a measure relative to shifts on vectors from a dense subspace. We will show, that on a Banach space X over $\mathbf{K} \supset \mathbf{Q_p}$ for each prime number p there is not a σ-additive $\mu \neq 0$ with values in $\mathbf{K_p}$ such that it is quasi-invariant relative to shifts from a dense subspace. Details can be lightly extracted from the results given above. Let on $(X, Bco(X))$ there exists such μ. With the help of suitable compact operators a cylindrical measure on an algebra of cylindrical subsets of X generates quasi-invariant measures, so we can suppose that μ is quasi-invariant. Then it produces a sequence of a finite-dimensional distribution $\{\mu_{L_n} : n \in \mathbf{N}\}$ analogously to § 2 and § 3, where L_n are subspaces of X with dimensions over $\mathbf{K}$ equal to n. Each measure μ_{L_n} is σ-additive.

From the quasi-invariance of μ it follows, that L_n can be chosen such that μ_{L_n} are quasi-invariant relative to the entire L_n. But in view of Chapters 7-9 [Roo78] and [Sch84] for measures with values in $\mathbf{K_p}$ (see also Proposition 11 from § VII.1.9 [Bou63-69]) this means that μ_{L_n} is equivalent to the Haar measure on L_n with values in $\mathbf{K_p}$.

The space L_n as the additive group can be considered over $\mathbf{Q_p}$, moreover, for each continuous linear functional $\phi : \mathbf{K_p} \to \mathbf{Q_p}$ considered as the finite-dimensional Banach space over $\mathbf{Q_p}$ the measure $\phi \circ \mu_{L_n}(*)$ is non-trivial for some ϕ. Consequently, on L_n there would be the Haar measure with values in $\mathbf{Q_p}$, but this is impossible due to Chapter 9 in [Roo78], since L_n is not the p-free group. We get the contradiction, that is, such μ does not exist.

3.22. Theorem. *Let A be a complete normed algebra over the local field $\mathbf{K}$. If a nontrivial $\mathbf{K_s}$-valued measure μ on $Bco(A)$ is quasi-invariant relative to dense subalgebra A' (relative to linear shifts and left (or right) multiplication), then A is finite dimensional over $\mathbf{K}$.*

3.23. Theorem. *Let A be a Banach space over the local field $\mathbf{K}$. If μ is a non-trivial $\mathbf{K_s}$-valued measure on $Bco(A)$ quasi-invariant relative to shifts from a dense $\mathbf{K}$-linear subspace L', then there exists a nontrivial topological group G of $\mathbf{K}$-linear automorphisms of A such that μ is also quasi-invariant relative to G.*

Proof. These statements of 3.22, 3.23 follow from Theorem 3.19 and Lemma 3.18 analogously to § § I.3.28, I.3.29 above.

2.4. Pseudo-differentiable $\mathbf{K_s}$-Valued Measures

4.1. Note and Definition. A function $f : \mathbf{K} \to \mathbf{U_s}$ is called pseudo-differentiable of order b, if there exists the following integral:

$$PD(b, f(x)) := \int_{\mathbf{K}} [(f(x) - f(y)) \times g(x,y,b)] dv(y).$$

We introduce the following notation $PD_c(b, f(x))$ for such integral by $B(\mathbf{K}, 0, 1)$ instead of the entire $\mathbf{K}$. Where $g(x,y,b) := s^{(-1-b)\times ord_p(x-y)}$ with the corresponding Haar measure v with values in $\mathbf{K_s}$, where $b \in \mathbf{C_s}$ and $|x|_{\mathbf{K}} = p^{-ord_p(x)}$, $\mathbf{C_s}$ denotes the field of complex numbers with the non-Archimedean normalization extending that of $\mathbf{Q_s}$, $\mathbf{U_s}$ is a spherically complete field with a normalization group $\Gamma_{\mathbf{U_s}} := \{|x| : 0 \neq x \in \mathbf{U_s}\} = (0, \infty) \subset \mathbf{R}$ such that $\mathbf{C_s} \subset \mathbf{U_s}$, $0 < s$ is a prime number (see also [Dia84, Roo78, Sch84, Wei73]). For each $\gamma \in (0, \infty)$ there exists $\alpha = \log_s(\gamma) \in \mathbf{R}$, $\Gamma_{\mathbf{U_s}} = (0, \infty)$, hence $s^\alpha \in \mathbf{U_s}$ is defined for each $\alpha \in \mathbf{R}$, where $log_s(\gamma) = ln(\gamma)/ln(s)$, $\ln : (0, \infty) \to \mathbf{R}$ is the natural logarithmic function such that $\ln(e) = 1$. The function $s^{\alpha+i\beta} =: \xi(\alpha, \beta)$ with α and $\beta \in \mathbf{R}$ is defined due to the algebraic isomorphism of $\mathbf{C_s}$ with $\mathbf{C}$ (see also [Kob77]) in the following manner. Put $s^{\alpha+i\beta} := s^{\alpha}(s^i)^{\beta}$ and choose as s^i a marked number in $\mathbf{U_s}$ such that $s^i := (EXP_s(i))^{\ln s}$, where $EXP_s : \mathbf{C_s} \to \mathbf{C_s^+}$ is the exponential function, $\mathbf{C_s^+} := \{x \in \mathbf{C_s} : |x-1|_s < 1\}$ (see Proposition 45.6 [Sch84]). Therefore, $|EXP_s(i) - 1|_s < 1$, hence $|EXP_s(i)|_s = 1$ and inevitably $|s^i|_s = 1$. Therefore, $|s^{\alpha+i\beta}|_s = s^{-\alpha}$ for each α and $\beta \in \mathbf{R}$, where $|*|_s$ is the extension of the normalization from $\mathbf{Q_s}$ on $\mathbf{U_s}$, consequently, $s^x \in \mathbf{U_s}$ is defined for each $x \in \mathbf{C_s}$.

A quasi-invariant measure μ on X is called pseudo-differentiable for $b \in \mathbf{C_s}$, if there exists $PD(b, g(x))$ for $g(x) := \mu(-xz + S)$ for each $S \in Bco(X)$ $\|S\|_\mu < \infty$ and each $z \in J^b_\mu$, where J^b_μ is a $\mathbf{K}$-linear subspace dense in X. For a fixed $z \in X$ such measure is called pseudo-differentiable along z.

For a one-parameter subfamily of operators $B(\mathbf{K}, 0, 1) \ni t \mapsto U_t : X \to X$ quasi-invariant measure μ is called pseudo-differentiable for $b \in \mathbf{C_s}$, if for each S the same as above there exists $PD_c(b, g(t))$ for a function $g(t) := \mu(U_t^{-1}(S)$, where X may be also a topological group G with a measure quasi-invariant relative to a dense subgroup G' (see [Lud99t, Lud98s, Lud00a]).

4.2. Let μ, X, and ρ be the same as in Theorem 3.15 and F be a non-Archimedean Fourier transform defined in [VVZ94, Roo78].

Theorems. (1) *$g(t) := \rho(z + tw, x) j(t) \in L(\mathbf{K}, \nu, \mathbf{K_s}) =: V$ for μ and the Haar measure ν with values in $\mathbf{K_s}$, where z and $w \in J_\mu$, $t \in \mathbf{K}$, $j(t)$ is the characteristic function of a compact subset $W \subset \mathbf{K}$. In general, may be $k(t) := \rho(z + tw, x) \notin V$.*

(2) *Let $g(t) = \rho(z + tw, x) j(t)$ with clopen subsets W in $\mathbf{K}$. Then there are μ, for which there exists $PD(b, g(t))$ for each $b \in \mathbf{C_s}$. If $g(t) = \rho(z + tw, x)$, then there are probability measures μ, for which there exists $PD(b, g(t))$ for each $b \in \mathbf{C_s}$ with $0 < Re(b)$ or $b = 0$.*

(3) *Let $S \in Af(X, y)$, $\|S\|_\mu < \infty$, then for each $b \in U := \{b' : Re\, b' > 0$ or $b' = 0\}$ there is a pseudo-differentiable quasi-invariant measure μ .*

Proof. We consider the following additive compact subgroup $G_T := \{x \in X | \|x(j)\| \le p^{k(j)}$ for each $j \in \mathbf{N}\}$ in X, where $T = diag\{d(j) \in K : |d(j)| = p^{-k(j)}$ for each $j \in \mathbf{N}\}$ is a compact diagonal operator. Then μ from Theorem 3.15 is quasi-invariant relative to the following additive subgroup $S_T := G_T + H$, where $H := span_{\mathbf{K}}\{e(j) : j \in \mathbf{N}\}$. The rest of the proof is analogous to that of § I.4.2.

4.2.1. Theorem. *Let X be a Banach space of separable type over a locally compact non-Archimedean field $\mathbf{K}$ and J be a dense proper $\mathbf{K}$-linear subspace in X such that the embedding operator $T : J \hookrightarrow X$ is compact and nondegenerate, $ker(T) = \{0\}$, $b \in \mathbf{C}$. Then a set $\mathcal{P}_b(X, J)$ of probability $\mathbf{K_s}$-valued measures μ on $Bco(X)$ quasi-invariant and pseudo-differentiable of order b relative to J is of cardinality $card(\mathbf{K_s})^{\mathsf{c}}$. If J', $J' \subset J$, is also a dense $\mathbf{K}$-linear subspace in X, then $\mathcal{P}_b(X, J') \supset \mathcal{P}_b(X, J)$.*

Proof. As in § I.3.20.1 choose for T an orthonormal base in X in which T is diagonal and X is isomorphic with c_0 over $\mathbf{K}$ such that in its standard base $\{e_j : j \in \mathbf{N}\}$ the operator T is characterized by $Te_j = a_j e_j$, $0 \ne a_j \in \mathbf{K}$ for each $j \in \mathbf{N}$, $\lim_{j\to\infty} a_j = 0$ (see Appendix A).

Take g_n from § 3.15.1, where $g_n \in L(\mathbf{K}, Bf(\mathbf{K}), w'(dx/a_n), \mathbf{K_s})$, satisfy conditions there and such that there exists $\lim_{m\to\infty} PD(b, \prod_{n=1}^m g_n(xz)) \in L(X, Bco(X), \nu, \mathbf{F})$ by the variable x for each $z \in J$, where $x \in \mathbf{K}$, $\mathbf{K_s} \cup \mathbf{C_s} \subset \mathbf{F}$, $\mathbf{F}$ is a non-Archimedean field. Evidently, $\mathcal{P}_b(X, J) \subset \mathcal{M}(X, J)$. The family of such sequences of functions $\{g_n : n \in \mathbf{N}\}$ has the cardinality $card(\mathbf{K_s})^{\mathsf{c}}$, since in $L(\nu)$ the subspace of step functions is dense and the condition of pseudo-differentiability is the integral convergence condition (see § § 4.1 and 4.2).

4.3. Let X be a Banach space over $\mathbf{K}$, $b_0 \in \mathbf{R}$ or $b_0 = +\infty$ and suppose that the following conditions are satisfied:

(1) $T : X \to X$ is a compact operator with $ker(T) = \{0\}$;

(2) a mapping $\tilde{F}$ from $B(\mathbf{K}, 0, 1)$ to $C_T(X) := \{U : U \in C^1(X, X)$ and $(U'(x) - I)$ is a compact operator for each $x \in X$, there is U^{-1} satisfying the same conditions as $U\}$ is given;

(3) $\tilde{F}(t) = U_t(x)$ and $\Phi^1 U_t(x + h, x)$ are continuous by t, that is, $\tilde{F} \in C^1(B(\mathbf{K}, 0, 1), C_T(X))$;

(4) there is $c > 0$ such that $\|U_t(x) - U_s(x)\| \le \|Tx\|$ for each $x \in X$ and $|t - s| < c$;

(5) for each $R > 0$ there is a finite-dimensional over $\mathbf{K}$ subspace $H \subset X$ and $c' > 0$ such that $\|U_t(x) - U_s(x)\| \le \|Tx\|/R$ for each $x \in X \ominus H$ and $|t - s| < c'$ with $(3-5)$ satisfying also for U_t^{-1}.

Theorem. *On X there are probability quasi-invariant measures μ which are pseudo-differentiable for each $b \in \mathbf{C_s}$ with $\mathbf{R} \ni Re(b) \le b_0$ relative to a family U_t, where μ are with values in $\mathbf{K_s}$.*

Proof. From Conditions (2,3) it follows that there is $c > 0$ such that $|det(U_t'(x))| = |det(U_s'(x))|$ in $L(\mu)$ by $x \in X$ and all $|t - s| < c$, where quasi-invariant and pseudo-differentiable measures μ on X relative to S_T may be constructed as in the proof of Theorems 3.15 and 4.2. The final part of the proof is analogous to that of § I.4.3.

4.4. Let X be a Banach space over $\mathbf{K}$, μ be a probability quasi-invariant measure $\mu : Bco(X) \to \mathbf{K_s}$, that is pseudo-differentiable for a given b with $Re(b) > 0$, $C_b(X)$ be a space of continuous bounded functions $f : X \to \mathbf{K_s}$ with $\|f\| := \sup_{x \in X} |f(x)|$.

Theorem. *For each $a \in J_\mu$ and $f \in C_b(X)$ is defined the following integral:*

$$(i)\ l(f) = \int_{\mathbf{K}} [\int_X f(x)[\mu(-\lambda a + dx) - \mu(dx)] g(\lambda, 0, b) v(d\lambda)$$

and there exists a measure $\nu : Bco(X) \to \mathbf{C_s}$ with a bounded variation, moreover, for $b \in \mathbf{R}$ this ν is a mapping from from $Bco(X)$ into $\mathbf{K_s}$, such that

$$(ii)\ l(f) = \int_X f(x) \nu(dx),$$

where v is the Haar measure on $\mathbf{K}$ with values in $\mathbf{Q_s}$, moreover, ν is independent from f and may be dependent on $a \in J_\mu$. We denote $\nu =: \tilde{D}_a^b \mu$.

Proof. From Definition 4.1 and the non-Archimedean analog of the Lebesgue convergence theorem it follows that there exists

$$\lim_{j \to \infty} \int_{\mathbf{K} \setminus B(\mathbf{K}, 0, p^{-j})} \left[\int_X (f(x + \lambda a) - f(x)) g(\lambda, 0, b) \mu(dx) \right] v(d\lambda) = l(f),$$

that is (i) exists. Let

$$(iii) \quad l_j(V, f) := \int_{\mathbf{K} \setminus B(\mathbf{K}, 0, p^{-j})} \left[\int_V f(x)(\mu(-\lambda a + dx) - \mu(dx)) g(\lambda, 0, b) \right] v(d\lambda),$$

where $V \in Bco(X)$. Then due to construction of § 3.15 for each $c > 0$ there exists a compact $V_c \subset X$ with $\|X \setminus V_c\|_{\nu_\lambda} < c$ for each $|\lambda| > 0$, where

$$\nu_\lambda(A) := \int_{\mathbf{K}\setminus B(\mathbf{K},0,|\lambda|)} [\mu(-\lambda' a + A) - \mu(A)] g(\lambda', 0, b) \nu(d\lambda')$$

for $A \in Bco(X)$. The rest of the proof is analogous to that of § I.4.4.

4.5. Theorem. *Let X be a Banach space over $\mathbf{K}$, $|*| = mod_{\mathbf{K}}(*)$ with a probability quasi-invariant measure $\mu : Bco(X) \to \mathbf{K_s}$ and it is satisfied Condition 3.16.(i), suppose μ is pseudo-differentiable and*

(viii) $J_b\mu \subset T"J_\mu$, $(U_t : t \in B(\mathbf{K}, 0, 1))$ is a one-parameter family of operators such that Conditions 3.20$(i - vii)$ are satisfied with the substitution of J_μ onto J_μ^b uniformly by $t \in B(\mathbf{K}, 0, 1)$, $J_\mu \supset T'X$, where $T', T" : X \to X$ are compact operators, $ker(T') = ker(T") = 0$. Moreover, suppose that there are sequences

(ix) $[k(i, j)]$ and $[k'(i, j)]$ with $i, j \in \mathbf{N}$, $\lim_{i+j\to\infty} k(i, j) = \lim_{i+j\to\infty} k'(i, j) = -\infty$ and $n \in \mathbf{N}$ such that $|T"_{i,j} - \delta_{i,j}| < |T'_{i,j} - \delta_{i,j}| p^{k(i,j)}$, $|U_{i,j} - \delta_{i,j}| < |T"_{i,j} - \delta_{i,j}| p^{k'(i,j)}$ and $|(U^{-1})_{i,j} - \delta_{i,j}| < |T"_{i,j} - \delta_{i,j}| p^{k'(i,j)}$ for each $i + j > n$, where $U_{i,j} = \tilde{e}_i U(e_j)$, $(e_j : j)$ is orthonormal basis in X. Then for each $f \in C_b(X)$ is defined

$$(\alpha)\ l(f) = \int_{B(K,0,1)} \left[\int_X f(x)[\mu(U_t^{-1}(dx)) - \mu(dx)] \right] g(t, 0, b) \nu(dt)$$

and there exists a measure $\nu : Bco(X) \to \mathbf{C_s}$ with a bounded total variation [particularly, for $b \in \mathbf{R}$ it is such that $\nu : Bco(X) \to \mathbf{K_s}$] and

$$(\beta)\ l(f) = \int_X f(x) \nu(dx),$$

where ν is independent from f and may be dependent on $(U_t : t)$, $\nu =: \tilde{D}^b_{U_*}\mu$.

Proof. From the proof of Theorem 3.20 it follows that there exists a sequence $U_t^{(q)}$ of polygonal operators converging uniformly by $t \in B(\mathbf{K}, 0, 1)$ to U_t and equicontinuously by indices of matrix elements in $L(\mu)$. Then there exists $\lim_{q\to\infty} \lim_{j\to\infty} \int_{B(\mathbf{K},0,1)\setminus B(\mathbf{K},0,p^{-j}} [\int_X f(U_t^{-1}(x)) - f(x)] g(t, 0, b) \mu(dx)] \nu(dt)$ for each $f \in C_b(X)$. From conditions $(viii, ix)$, the Fubini and Lebesgue theorems it follows that for $\nu_\lambda := \int_{B(\mathbf{K},0,1)\setminus B(\mathbf{K},0,|\lambda|)} [\mu(U_t^{-1}(A)) - \mu(A)] g(t, 0, b) \nu(dt)$ for $A \in Bco(X)$ for each $c > 0$ there exists a compact $V_c \subset X$ and $\delta > 0$ such that $\|X \setminus V_c\| < c$. Indeed, V_c and $\delta > 0$ may be chosen due to pseudo-differentiability of μ, § § 2.30, 3.18, Formula (i), 3.16.(i) and due to continuity and boundednessy (on $B(\mathbf{K}, 0, 1) \ni t$) of $|det\, U_t'(U_t^{-1})(x))|_{\mathbf{K}}$ satisfying the following conditions $U_t^{-1}(V_c) \subset V_c$ and $\|(X \setminus V_c) \triangle (U_t^{-1}(X \setminus V_c))\|_\mu = 0$ for each $|t| < \delta$, since $V_c = Y(j) \cap V_c$ are compact for every j. Repeating proofs 3.20 and 4.4 with the use of Lemma I.2.5 for the family $(U_t : t)$ we get formulas (α, β).

2.5. Convergence of $\mathbf{K_s}$-Valued Measures

5.1. Definitions, notes and notations. Let S be a normal topological group with the small inductive dimension $ind(S) = 0$, S' be a dense subgroup, suppose their topologies are τ and τ' correspondingly, $\tau' \supset \tau|_{S'}$. Let G be an additive Hausdorff left-R-module, where R is a topological ring, $\mathsf{R} \supset Bco(S)$ be a a ring $\mathsf{R} \supset Bco(S)$ for $\mathbf{K_s}$-valued measures, $\mathsf{M}(\mathsf{R}, G)$ be a family of measures with values in G, $\mathsf{L}(\mathsf{R}, G, R)$ be a family of quasi-invariant measure $\mu : \mathsf{R} \to G$ with $\rho_\mu(g,x) \times \mu(dx) := \mu^{g^{-1}}(dx) =: \mu(gdx)$, $R \times G \to G$ be a continuous left action of R on G such that $\rho_\mu(gh,x) = \rho_\mu(g,hx)\rho_\mu(h,x)$ for each $g,h \in S'$ and $x \in S$. Particularly, $1 = \rho_\mu(g,g^{-1}x)\rho_\mu(g^{-1},x)$, that is, $\rho_\mu(g,x) \in R_o$, where R_o is a multiplicative subgroup of R. Moreover, $zy \in \mathsf{L}$ for $z \in R_0$ with $\rho_{z\mu}(g,x) = z\rho_\mu(g,x)z^{-1}$ and $z \neq 0$. We suppose that topological characters and weights S and S' are countable and each open W in S' is pre-compact in S. Let $\mathbf{P}$" be a family of pseudo-metrics in G generating the initial uniformity such that for each $c > 0$ and $d \in \mathbf{P}$" and $\{U_n \in \mathsf{R} : n \in \mathbf{N}\}$ with $\cap\{U_n : n \in \mathbf{N}\} = \{x\}$ there is $m \in \mathbf{N}$ such that $d(\mu^g(U_n), \rho_\mu(g,x)\mu(U_n)) < cd(\mu(U_n),0)$ for each $n > m$, in addition, a limit ρ is independent μ-a.e. on the choice of $\{U_n : n\}$ for each $x \in S$ and $g \in S'$. Consider a sub-ring $R' \subset \mathsf{R}$, $R' \supset Bco(S)$ such that $\cup\{A_n : n = 1,\dots,N\} \in R'$ for $A_n \in R'$ with $N \in \mathbf{N}$ and $S'R' = R'$. Then $\mathsf{L}(\mathsf{R}, G, R; R') := \{(\mu, \rho_\mu(*,*)) \in \mathsf{L}(\mathsf{R}, G, R) : \mu - R' -$ is regular and for each $s \in S$ there are $A_n \in R', n \in \mathbf{N}$ with $s = \cap(A_n : n), \{s\} \in R'\}$.

For pseudo-differentiable measures μ let $S" \subset S'$, $S"$ be a dense subgroup in S, $\tau'|S"$ is not stronger than $\tau"$ on $S"$ and there exists a neighborhood $\tau" \ni W" \ni e$ in which are dense elements lying on one-parameter subgroups $(U_t : t \in B(\mathbf{K},0,1)))$. We suppose that μ is induced from the Banach space X over $\mathbf{K}$ due to a local homeomorphism of neighborhoods of e in S and 0 in X as for the case of groups of diffeomorphisms [Lud96] such that is accomplished Theorem 4.5 for each $U_* \subset S"$ inducing the corresponding transformations on X. In the following case $S = X$ we consider $S' = J_\mu$ and $S" = J_\mu^b$ with $Re(b) > 0$ such that $M_\mu \supset J_\mu \subset (T_\mu X)^\sim$, $J_\mu^b \subset (T_\mu^{(b)} X)^\sim$ with compact operators T_μ and $T_\mu^{(b)}$, $ker(T_\mu) = ker(T_\mu^{(b)}) = 0$ and norms induced by the Minkowski functional P_E for $E = T_\mu B(X,0,1)$ and $E = T_\mu^{(b)} B(X,0,1)$ respectively. We suppose further that for pseudo-differentiable measures G is equal to $\mathbf{C_s} \vee \mathbf{K_s}$. We denote $P(\mathsf{R}, G, R, U_*; R') := [(\mu, \rho_\mu, \eta_\mu) : (\mu, \rho_\mu) \in \mathsf{L}(\mathsf{R}, G, R; R'), \mu$ is pseudo-differentiable and $\eta_\mu(t, U_*, A) \in L(\mathbf{K}, v, \mathbf{C_s})]$, where $\eta_\mu(t, U_*, A) = j(t)g(t,0,b)[\mu^h(U_t^{-1}(A) - \mu^h(A)]$, $j(t) = 1$ for each $t \in \mathbf{K}$ for $S = X$; $j(t) = 1$ for $t \in B(\mathbf{K},0,1)$, $j(t) = 0$ for $|t|_{\mathbf{K}} > 1$ for a topological group S that is not a Banach space X over $\mathbf{K}$, v is the Haar measure on $\mathbf{K}$ with values in $\mathbf{Q_s}$, $(U_t : t \in B(\mathbf{K},0,1))$ is an arbitrary one-parameter subgroup. On these spaces L (or P) the additional conditions are imposed:

(a) for each neighborhood (implying that it is open) $U \ni 0 \in G$ there exists a neighborhood $S \supset V \ni e$ and a compact subset V_U, $e \in V_U \subset V$, with $\mu(B) \in U$ (or in addition $\tilde{D}^b_{U_*}\mu(B) \in U$) for each B, $\mathsf{R} \ni B \in Bco(S \setminus V_U)$;

(b) for a given U and a neighborhood $R \supset D \ni 0$ there exists a neighborhood W, $S' \supset W \ni e$, (pseudo)metric $d \in P"$ and $c > 0$ such that $\rho_\mu(g,x) - \rho_\mu(h,x') \in D$ (or $\tilde{D}^b_{U_*}(\mu^g - \mu^h)(A) \in U$ for $A \in Bco(V_U)$ in addition for P) whilst $g,h \in W$, $x,x' \in V_U$, $d(x,x') < c$, where (a,b) is satisfied for all $(\mu, \rho_\mu) \in \mathsf{L}$ (or $(\mu, \rho_\mu, \eta_\mu) \in P$) equicontinuously in (a) on $V \ni U_t, U_t^{-1}$ and in (b) on W and on each V_U for $\rho_\mu(g,x) - \rho_\mu(h,x')$ and $\tilde{D}^b_{U_*}(\mu^g - \mu^h)(A)$.

These conditions are justified, since due to Theorems 3.15, 3.19, 4.3 and 4.5 there exists a subspace $Z"$ dense in Z' such that for each $\varepsilon > 0$ and each $\infty > R > 0$ there are $r > 0$ and $\delta > 0$ with $|\rho_\nu(g,x) - \rho_\nu(h,y)| < \varepsilon$ for each $\|g-h\|_{Z"} + \|x-y\|_Z < \delta$, $g,h \in B(Z",0,r)$, $x,y \in B(Z,0,R)$, where $Z"$ is the Banach space over $\mathbf{K}$. For a group of diffeomorphisms of a non-Archimedean Banach manifold we have an analogous continuity of ρ_μ for a subgroup $G"$ of the entire group G (see [Lud96, Lud99t, Lud00a, Lud02b]). By M_o we denote a subspace in M, satisfying (a). Henceforth, we imply that R' contains all closed subsets from S belonging to R, where G and R are complete.

For $\mu : Bco(S) \to G$ by $L(S,\mu,G)$ we denote the completion of a space of continuous $f : S \to G$ such that $\|f\|_d := \sup_{h\in C_b(S,G)} d(\int_S f(x)h(x)\,\mu(dx),0) < \infty$ for each $d \in P$", where $C_b(S,G)$ is a space of continuous bounded functions $h : S \to G$. We suppose that for each sequence $(f_n : n) \subset L(S,\mu,G)$ for which $g \in L(S,\mu,G)$ exists with $d(f_n(x),0) \le d(g(x),0)$ for every $d \in P$", x and n, that f_n converges uniformly on each compact subset $V \subset S$ with $\|V\|_\mu > 0$ and the following is satisfied: $f \in L(S,\mu,G)$, $\lim_n \|f_n - f\|_d = 0$ for each $d \in P$" and $\int_S f(x)\mu(dx) = \lim_n \int_S f_n(x)\mu(dx)$. In the case $G = \mathbf{K_s}$ it coincides with $L(S,\mu,\mathbf{K_s})$, hence this supposition is the Lebesgue theorem. By $Y(v)$ we denote $L(\mathbf{K},v,\mathbf{C_s})$.

Now we may define topologies and uniformities with the help of corresponding bases (see below) on $\mathsf{L} \subset G^{\mathsf{R}} \times R_o^{S'\times S} =: Y$ (or $P \subset G^{\mathsf{R}} \times R_o^{S'\times S} \times G^{S'\times K\times \mathsf{R}} =: Y$, $R_o \subset R\setminus\{0\}$. There are the natural projections $\pi : \mathsf{L}\ (\vee\ P) \to M_o$, $\pi(\mu,\rho_\mu(*,*)\ (\vee\ ,\eta_\mu)) = \mu$, $\xi : \mathsf{L}\ (\vee\ P) \to R^{S'\times S}$, $\xi(\mu,\rho_\mu,(\vee\ \eta_\mu)) = \rho_\mu$, $\zeta : P \to G^{S'\times K\times \mathsf{R}}$, $\zeta(\mu,\rho_\mu,\eta_\mu) = \eta_\mu$. Let H be a filter on L or P, $U = U' \times U$" or $U = U' \times U" \times U"'$, U' and U" be elements of uniformities on G, R and $Y(v)$ correspondingly, $\tau' \ni W \ni e$, $\tau \ni V \supset V_{U'} \ni e$, $V_{U'}$ is compact. By $[\mu]$ we denote (μ,ρ_μ) for L or (μ,ρ_μ,η_μ) for P, $\Omega := \mathsf{L} \vee P$, $[\mu](A,W,V) := [\mu^g(A),\rho_\mu(g,x),\vee\,\eta_{\mu^g}(t,U_*,A)|$ $g\in W, x\in V, \vee\ t \in K]$. We consider $\mathsf{A} \subset \mathsf{R}$, then

$$\mathsf{W}(\mathsf{A},W,V_{U'};U) := \{([\mu],[\nu]) \in \Omega^2 | ([\mu],[\nu])(A,W,V_{U'}) \subset U\}; \qquad (1)$$

$$\mathsf{W}(\mathsf{S};U) := \{([\mu],[\nu]) \in \Omega^2 | \{(B,g,x) : ([\mu],[\nu])(B,g,x)) \in U\} \in \mathsf{S}\}, \qquad (2)$$

where S is a filter on $\mathsf{R} \times S' \times S^c$, S^c is a family of compact subsets $V' \ni e$.

$$\mathsf{W}(\mathsf{F},W,V;U) := \{([\mu],[\nu]) \in \Omega^2 | \{B : ([\mu],[\nu])(B,g,x) \in U, g \in W, x \in V\} \in \mathsf{F}\}, \qquad (3)$$

where F is a filter on R (see also § 2.1 and 4.1[Con84]);

$$\mathsf{W}(\mathsf{A},\mathsf{G};U) := \{([\mu],[\nu]) \in \Omega^2 |\ \{(g,x) : ([\mu],[\nu])(B,g,x) \in U, B \in \mathsf{A}\} \in \mathsf{G}\}, \qquad (4)$$

where G is a filter on $S' \times S^c$; suppose $\mathsf{U} \subset \mathsf{R} \times \tau'_e \times S^c$, Φ is a family of filters on $\mathsf{R} \times S' \times S^c$ or $\mathsf{R} \times S' \times S^c \times Y(v)$ (generated by products of filters $\Phi_{\mathsf{R}} \times \Phi_{S'} \times \Phi_{S^c}$ on the corresponding spaces), U' be a uniformity on (G,R) or $(G,R,Y(v))$, $\mathsf{F} \subset Y$. A family of finite intersections of sets $\mathsf{W}(A,U) \cap (\mathsf{F}\times\mathsf{F})$ (see (1)), where $(A,U) \in \mathsf{U} \times \mathsf{U}'$ (or $\mathsf{W}(F,U) \cap (\mathsf{F}\times\mathsf{F})$ (see (2)), where $(F,U) \in (\Phi \times \mathsf{U}')$ generate by the definition a base of uniformity of U-convergence (Φ-convergence respectively) on F and generate the corresponding topologies. For these uniformities are used notations

(i) F_{U} and F_Φ; $\mathsf{F}_{\mathsf{R}\times W\times V}$ is for F with the uniformity of uniform convergence

on $\mathsf{R} \times W \times V$, where $W \in \tau'_e$, $V \in S^c$, analogously for the entire space Y;

(ii) F_A denotes the uniformity (or topology) of pointwise convergence for

$A \subset \mathsf{R} \times \tau'_e \times S^c =: Z$, for $A = Z$ we omit the index (see formula (1)). Henceforward, we use H' instead of H in 4.1.24[Con84], that is, $\mathsf{H}'(A,\tilde{\mathsf{R}})$-filter on R generated by the base $[(L \in \mathsf{R} : L \subset A\setminus K') : K' \in \tilde{\mathsf{R}},\ K' \subset A]$, where $\tilde{\mathsf{R}} \subset \mathsf{R}$ and $\tilde{\mathsf{R}}$ is closed relative to the finite unions.

For example, let S be a locally **K**-convex space, S' be a dense subspace, G be a locally **L**-convex space, where $\mathbf{K},\mathbf{L}$ are fields, $R = B(G)$ be a space of bounded linear operators on G, $R_o = GL(G)$ be a multiplicative group of invertible linear operators. Then others possibilities are: $S = X$ be a Banach space over **K**, $S' = J_\mu$, $S" = J^b_\mu$ as above; $S = G(t)$, $S' \supset S"$ are dense subgroups, $G = R$ be the field $\mathbf{K_s}$ ($s \ne p$), M be an analytic Banach manifold over $\mathbf{K} \supset \mathbf{Q_p}$ (see [Lud96]). The rest of the necessary standard definitions are recalled further when they are used.

5.2. Lemma. *Let* R *be a quasi-*δ*-ring with the weakest uniformity in which each* $\mu \in \mathsf{M}$ *is uniformly continuous and* $\Phi \subset \hat{\Phi}_C(\mathsf{R}, S' \times S^c)$. *Then* $\mathsf{L}(\mathsf{R}, G, R, R')_\Phi$ *(or* $P(\mathsf{R}, G, R, U_*; R')_\Phi)$*) is a topological space on which* R_o *acts continuously from the right.*

Proof. It is analogous to that of § I.5.2 using Definition 4.1 for pseudo-differentiable f.

5.3. Proposition. (1). *Let* T *be a* $\hat{\Phi}_4$*-filter on* $\mathsf{M_o}(\mathsf{R}, G; R')$, $\{A_n\}$ *be a disjoint* $\Theta(\mathsf{R})$*-sequence,* Σ *be the elementary filter on* R *generated by* $\{A_n : n \in \mathbf{N}\}$ *and* $\phi : \mathsf{M_o} \times \mathsf{R} \to G$ *with* $\phi(\mu, A) = \mu(A)$. *Then* $\phi(\mathsf{T} \times \Sigma)$ *converges to* 0.

(2). *Moreover, let* U *be a base of neighborhoods of* $e \in S'$, $\phi : \mathsf{L} \to G \times R$, $\phi(\mu, A, g) := (\mu^g(A), \rho_\mu(g, x))$, *where* $x \in A$. *Then* $(0, 1) \in \lim \phi(\mathsf{T} \times \Sigma \times \mathsf{U})$.

(3). *If* T *is a* Φ_4*-filter on* $P(\mathsf{R}, G, R, U_*; R')$,

$$\psi(\mu, B, g, t, U_*) = [\mu(B); \rho_\mu(g, x); \eta_{\mu^g}(t, U_*, B)],$$

then $(0, 1, 0) \in \lim \psi(\mathsf{T} \times \Sigma \times \mathsf{U})$ *for each given* $U_* \in S"$, *where* Σ *and* U *as in* $(1, 2)$.

Proof. The proof is analogous to that of § I.5.3 with the use of the Lebesgue convergence theorem.

5.4. Proposition. *Let* H *be a* $\hat{\Phi}_4$*-filter on* L *(or* P*) with the topology* F *(see* 5.1*(ii)),* $A \in \mathsf{R}$, $\tau_G \ni U \ni 0$, $\mathsf{H}'(A, R') \in \Psi_f(\mathsf{R})$. *Then there are* $L \in \mathsf{H}$, $\tilde{K} \in R'$ *and an element of the uniformity* U *for* $\mathsf{L}_{R'}$ *or* $P_{R'}$ *such that* $\tilde{\mathbf{K}} \subset A$, $L = [(\mu, \rho_\mu(g, x)) : M := \pi_{\mathsf{M_o}}(L) \ni \mu, \pi_{\tau'_e}(L) =: W \ni g$ *(or* $(\mu, \rho_\mu, \eta_\mu(*, *, U_*))$ *and additionally* $\tilde{D}^b_{U_*}\mu = PD(b, \eta_\mu))]$, $e \in W \in \tau'$, $\mu^g(B) - \nu^h(C) \in U$ *(or in addition* $(\tilde{D}^b_{U_*}\mu^g(B)) - (\tilde{D}^b_{U_*}\nu^h(C)) \in U$*) for* $\tilde{K} \subset B \subset A$, $\tilde{K} \subset C \subset A$ *for each* $([\mu], [\nu]) \in \bar{\bar{L}}^2 \cap \mathsf{U}$, *where* $\bar{\bar{L}} := cl(L, \mathsf{L}_{R'})$ *(or* $cl(L, P_{R'})$*),* $\pi_{\mathsf{M_o}}$ *is a projector from* L *into* $\mathsf{M_o}$.

Proof. Repeating the proof of § I.5.4 we get $\mu^g(B) - \mu(B) \in U'$, $\nu^h(C) - \nu(C) \in U'$ and for $3U' \subset U$ we get 5.4 for L. From Theorems 4.4 and 4.5, §5.1, the Egorov conditions and the Lebesgue theorem we get 5.4 for P, since μ are probability measures and $\mathsf{L}_{R'}$ (or $P_{R'}$) correspond to uniformity from § 5.1(ii) with $A = R' \times \tau'_e \times S^c$. Indeed, $\mu^g(A) - \nu^h(A) = (\mu^g(A) - \mu^g(V_{U'})) + (\mu^g(V_{U'}) - \nu^h(V_{U'})) + (\nu^h(V_{U'}) - \nu^h(A))$, $\mu^g(A) = \int_A \rho_\mu(g, x)\mu(dx)$ for each $A \in Bco(S)$, for each $\tau_G \ni U' \ni 0$ there exists a compact subset $V'_U \subset A$ with $\mu^g(B) \in U'$ for each $B \in Bf(A \setminus V_{U'}) \cap Bco(S)$ and the same for ν^h (due to the condition in § 5.1 that R' contains $Bco(S)$). At first we can consider $A \in Bco(S)$, then use R'-regularity of measures and $\sigma R' \supset Bco(S)$.

From the separability of S, S' and the equality of their topological weights to $\aleph_0$, restrictions 5.1.(a,b) it follows that there exists a sequence of partitions $Z_n = [(x_m, A_m) : m, x_m \in A_m]$ for each $A \in Bco(S)$, $A_i \cap A_j = \emptyset$ for each $i \ne j$, $\bigcup_m A_m = A$, $A_m \in Bco(S)$, such that $\lim_{n\to\infty}(\mu^g(A) - \sum_j \rho_\mu(g, x_j)\mu(A_j)) = 0$ and the same for ν, moreover, for $V_{U'}$ each Z_n may be chosen finite. Then there exists $W \in \tau'_e$ with $W \times (S \setminus V^2) \subset (S \setminus V)$, $\tau_e \ni V \subset V^2$, $\nu^g(B)$ and $\mu^g(B) \in U'$ for each $B \in Bf(S \setminus V^2) \cap Bco(S)$ (for $G = \mathbf{K_s}$ respectively) and

$g \in W$ (see 5.1.(a)). Then from $A = [A \cap (S \setminus V^2)] \cup [A \cap V^2]$ and the existence of compact $V'_{U'} \subset V$ with $\mu(E) \in U'$ for each $E \in Bf(V \setminus V'_{U'}) \cap Bco(S)$ and the same for ν, moreover, $(V'_{U'})^2$ is also compact, it follows that $\mu^g(B) - \nu^h(C) \in U$ for $9U' \subset U$, since $R' \supset Bco(S)$, where W satisfies the following condition $\mu^g(V'_{U'}) - \nu^h(V'_{U'}) \in U'$ for $V'_{U'} \subset V^2$ due to § 5.1.(b), $\mu(B) - \nu(C) \in U'$, $WV'_{U'} \subset (V'_{U'})^2$ due to pre-compactness of W in S. Since pseudo-differentiable measures are also quasi-invariant, hence for them 5.4 is true.

Now let $[\mu] \in \lim H$, $A \in Bco(S)$, then $\eta_\mu \in \lim \zeta(H)$ in $Y(v)$ and there exists a sequence η_{μ_n} such that $\int_{\mathbf{K}} \eta_{\mu_n}(\lambda, U_*, A) v(d\lambda) = \tilde{D}^b_{U_*}\mu_n(A)$ due to § § 4.4 or 5.1 and $\lim_{n\to\infty} \tilde{D}^b_{U_*}\mu_n(A) = \int_{\mathbf{K}} \eta_\mu(\lambda, U_*, A) v(d\lambda) =: \kappa(A)$ due to the Lebesgue theorem. From $\eta_\mu(\lambda, U_*, A \cup B) = \eta(\lambda, U_*, A) + \eta(\lambda, U_*, B)$ for $A \cap B = \emptyset$, $B \in Bco(S)$ it follows that $\nu(A)$ is the measure on $Bco(S)$, moreover, $\kappa(A) = \tilde{D}^b_{U_*}\mu(A)$. Since $\mu^g(A) = \int_A \rho_\mu(g,x)\mu(dx)$ for $A \in Bco(S)$ for $g \in S'$, then

$$\eta_{\mu^g}(\lambda, U_*, A) = j(\lambda) g(\lambda, 0, b)[\mu^g(A) - \mu^g(U_\lambda^{-1}A)]$$

$$= j(\lambda) g(\lambda, 0, b) \int_A \rho_\mu(g,x)[\mu(dx) - \mu^{U_\lambda}(dx)]$$

and in view of the Fubini theorem there exists

$$\tilde{D}^b_{U_*}\mu^g(A) = \int_A \left[\int_K \rho_\mu(g,x) j(\lambda) g(\lambda,0,b)[\mu(dx) - \mu^{U_\lambda}(dx)] \right] v(d\lambda)$$

, where $j(t) = 1$ for $S = X$ and $j(t)$ is the characteristic function of $B(\mathbf{K}, 0, 1)$ for S that is not the Banach space X. Then μ-a.e. $\tilde{D}^b_{U_*}\mu^g(dx)/\tilde{D}^b_{U_*}\mu(dx)$ coincides with $\rho_\mu(g,x)$ due to $5.1(a,b)$, hence, $(\tilde{D}^b_{U_*}\mu^g, \rho_{\mu^g})$ generate the Φ_4-filter in L arising from the $\hat{\Phi}_4$-filter in P. Then we estimate $\tilde{D}^b_{U_*}(\mu^g - \nu^h)(A)$ as above $\mu^g(A) - \nu^h(A)$. Therefore, we find for the Φ_4-filter corresponding L, since there exists $\delta > 0$ such that $U_\lambda \in W$ for each $|\lambda| < \delta$. For Φ_4-filter we use the corresponding finite intersections $W_1 \cap \cdots \cap W_n = W$, where W_j correspond to the Φ_4-filters H_j.

Note. The formulations and proofs of § § 5.5-5.10 (see Chapter I) are quite analogous for real-valued and $\mathbf{K_s}$-valued measures due to preceding results.

2.6. Measures with Particular Properties

1. Theorem. *Let X be a complete separable ultra-uniform space and let $\mathbf{K}$ be a locally compact field. Then for each marked $b \in \mathbf{C_s}$ there exists a nontrivial $\mathbf{F}$-valued measure μ on X which is a restriction of a measure ν in a measure space $(Y, Bco(Y), \nu) = \lim\{(Y_m, Bco(Y_m), \nu_m), \bar{f}^m_n, E\}$ on X and each ν_m is quasi-invariant and pseudo-differentiable for $b \in \mathbf{C_s}$ relative to a dense subspace Y'_m, where $Y_n := c_0(\mathbf{K}, \alpha_n)$, $\bar{f}^m_n : Y_m \to Y_n$ is a normal (that is, $\mathbf{K}$-simplicial non-expanding) mapping for each $m \geq n \in E$, $f^m_n|_{P_m} = \bar{f}^m_n$. Moreover, if X is not locally compact, then the family $\mathcal{F}$ of all such μ contains a subfamily $\mathcal{G}$ of pairwise orthogonal measures with the cardinality $card(\mathcal{G}) = card(\mathbf{F})^{\mathsf{c}}$, $\mathsf{c} := card(\mathbf{Q_p})$.*

Proof. Choose a polyhedral expansion of X in accordance with Theorem B.2.18. Let either $\mathbf{Q_p} \subset \mathbf{K}$ for $char(\mathbf{K}) = 0$ or $\mathbf{F_p}(\theta) \subset \mathbf{K}$ for $char(\mathbf{K}) = p$, $s \neq p$ are prime numbers, $\mathbf{Q_s} \subset \mathbf{F}$, where $\mathbf{F}$ is a non-Archimedean field complete relative to its uniformity. On each

X_n take a probability $\mathbf{F}$-valued measure ν_n such that $\|X_n \setminus P_n\|_{\nu_n} < \varepsilon_n$, $\sum_{n\in E} \varepsilon_n < 1/5$. In accordance with § 3.5.1 and § 4.2.1 each ν_n can be chosen quasi-invariant and pseudo-differentiable for $b \in \mathbf{C_s}$ relative to a dense $\mathbf{K}$-linear subspace Y'_n, since each normal mapping f_n^m has a normal extension on Y_m supplied with the uniform polyhedra structure.

Since E is countable and ordered, then a family ν_n can be chosen by transfinite induction consistent, that is, $\bar{f}_n^m(\nu_m) = \nu_n$ for each $m \geq n$ in E, $\bar{f}_n^m(Y'_m) = Y'_n$. Then $X = \lim\{P_m, f_n^m, E\} \hookrightarrow Y$. Since $\bar{f}_n^m$ are $\mathbf{K}$-linear, then $(\bar{f}_n^m)^{-1}(Bco(Y_n)) \subset Bco(Y_m)$ for each $m \geq n \in E$. Therefore, ν is correctly defined on the algebra $\bigcup_{n\in E} f_n^{-1}(Bco(Y_n))$ of subsets of Y, where $f_n : X \to X_n$ are $\mathbf{K}$-linear continuous epimorphisms. Since ν is nontrivial and $\|\nu\|$ is bounded by 1, then by the non-Archimedean analog of the Kolmogorov theorem 2.39 ν has an extension on the algebra $Bco(Y)$ and hence on its completion $Af(Y,\nu)$.

Put $Y' := \lim\{Y'_m, \bar{f}_n^m, E\}$. Then ν_m on Y_m is quasi-invariant and pseudo-differentiable for $b \in \mathbf{C_s}$ relative to Y'_m. From $\sum_n \varepsilon_n < 1/5$ it follows, that $1 \geq \|X\|_\mu \geq \prod_n (1-\varepsilon_n) > 1/2$, hence μ is nontrivial.

To prove the latter statement use the non-Archimedean analog of the Kakutani theorem for $\prod_n Y_n$ and then consider the embeddings $X \hookrightarrow Y \hookrightarrow \prod_n Y_n$ such that projection and subsequent restriction of the measure $\prod_n \nu_n$ on Y and X are nontrivial, which is possible due to the proof given above. If $\prod_n \nu_n$ and $\prod_n \nu'_n$ are orthogonal on $\prod_n Y_n$, then they give ν and ν' orthogonal on X.

2. Definitions and Notes. Let spaces X and Y be as in § 1.6.4. Consider a non-Archimedean field $\mathbf{F}$ such that $\mathbf{K_s} \subset \mathbf{F}$ and with the normalization group $\Gamma_{\mathbf{F}} = (0,\infty) \subset \mathbf{R}$ and $\mathbf{F}$ is complete relative to its uniformity (see [Dia84, Esc95]). Then a measure $\mu = \mu_{q,B,\gamma}$ on X with values in $\mathbf{K_s}$ is called a q-Gaussian measure, if its characteristic functional $\hat{\mu}$ with values in $\mathbf{F}$ has the form

$$\hat{\mu}(z) = s^{[B(v_q^s(z),v_q^s(z))]}\chi_\gamma(z)$$

on a dense $\mathbf{K}$-linear subspace $\mathsf{D}_{q,B,X}$ in X^* of all continuous $\mathbf{K}$-linear functionals $z : X \to \mathbf{K}$ of the form $z(x) = z_j(\phi_j(x))$ for each $x \in X$ with $v_q^s(z) \in \mathsf{D}_{B,Y}$, where B is a nonnegative definite bilinear $\mathbf{R}$-valued symmetric functional on a dense $\mathbf{R}$-linear subspace $\mathsf{D}_{B,Y}$ in Y^*, $B : \mathsf{D}_{B,Y}^2 \to \mathbf{R}$, $j \in \Upsilon$ may depend on z, $z_j : X_j \to \mathbf{K}$ is a continuous $\mathbf{K}$-linear functional such that $z_j = \sum_{k\in\alpha_j} e_j^k z_{k,j}$ is a countable convergent series such that $z_{k,j} \in \mathbf{K}$, e_j^k is a continuous $\mathbf{K}$-linear functional on X_j such that $e_j^k(e_{l,j}) = \delta_l^k$ is the Kroneker delta symbol, $e_{l,j}$ is the standard orthonormal (in the non-Archimedean sense) basis in $c_0(\alpha_j,\mathbf{K})$, $v_q^s(z) = v_q^s(z_j) := \{|s^{q\ ord_p(z_{k,j})/2}|_s : k \in \alpha_j\}$. It is supposed that z is such that $v_q^s(z) \in l_2(\alpha_j,\mathbf{R})$, where q is a positive constant, $\chi_\gamma(z) : X \to \mathbf{T_s}$ is a continuous character such that $\chi_\gamma(z) = \chi(z(\gamma))$, $\gamma \in X$, $\chi : \mathbf{K} \to \mathbf{T_s}$ is a nontrivial character of $\mathbf{K}$ as an additive group (see Chapter 9 in [Roo78] and § 2.5 above).

3. Proposition. *A q-Gaussian quasi-measure on an algebra of cylindrical subsets $\bigcup_j \pi_j^{-1}(\mathcal{R}_j)$, where X_j are finite-dimensional over* $\mathbf{K}$ *subspaces in X, is a measure on a covering ring $\mathcal{R}$ of subsets of X* (*see* § 2.36). *Moreover, a correlation operator B is of class L_1, that is, $Tr(B) < \infty$, if and only if each finite dimensional over* $\mathbf{K}$ *projection of μ is a q-Gaussian measure* (*see* §2.1).

Proof. From Definition 2 it follows, that each one dimensional over $\mathbf{K}$ projection $\mu_{x\mathbf{K}}$ of a measure μ satisfies Conditions 2.1.$(i-iii)$ the covering ring $Bco(\mathbf{K})$, where $0 \neq x = e_{k,l} \in X_l$. Therefore, μ is defined and finite additive on a cylindrical algebra

$\mathsf{U} := \bigcup_{k_1,\dots,k_n;l} \phi_l^{-1}[(\phi^l_{k_1,\dots,k_n})^{-1}(Bco(span_{\mathbf{K}}\{e_{k_1,l},\dots,e_{k_n,l}\}))]$,
where $\phi^l_{k_1,\dots,k_n} : X_l \to span_{\mathbf{K}}(e_{k_1,l},\dots,e_{k_n,l})$ is a projection. This means that μ is a bounded quasi-measure on U. Since $\hat{\mu}(0) = 1$, then $\mu(X) = 1$. The characteristic functional $\hat{\mu}$ satisfies Conditions 2.5.$(3,5)$. In view of the non-Archimedean analog of the Bochner-Kolmogorov theorem (see § 2.21 above) and Theorem 2.37 μ has an extension to a probability measure on a covering ring $\mathcal{R}$ of subsets of X containing U.

Suppose that B is of class L_1. Then $B(v_q(z),v_q(z))$ and hence $\hat{\mu}(z)$ is correctly defined for each $z \in \mathsf{D}_{q,B,X}$. The set $\mathsf{D}_{q,B,X}$ of functionals z on X from § 2 separates points of X. From Definition 2 it follows, that $\hat{\mu}(y)$ is continuous.

Consider a diagonal compact operator T in the standard orthonormal base, $Te_{k,l} = a_{k,l}e_{k,l}$, $\lim_{k+l\to\infty} a_{k,l} = 0$. Since B is continuous, then the corresponding to B correlation operator E is a bounded $\mathbf{K}$-linear operator on Y, $\|E\| < \infty$. For each $\varepsilon > 0$ there exist $\delta > 0$ and T such that $\max(1,\|E\|)\delta < \varepsilon$ and $|a_{k,l}| < \delta$ for each $k+l > N$, where N is a marked natural number, therefore, $\|E|_{span_{\mathbf{K}}\{e_{k,l}:k+l>N\}}\| < \varepsilon$. Hence for each $\varepsilon > 0$ there exists a compact operator T such that from $|\tilde{z}Tz| < 1$ it follows, $|\hat{\mu}(y) - \hat{\mu}(x)| < \varepsilon$ for each $x-y = z$, where $x,y,z \in Y^*$. Therefore, by Theorem 2.30 the characteristic functional $\hat{\mu}$ defines a probability Radon measure on $Bco(X)$.

Vice versa suppose that each finite dimensional over $\mathbf{K}$ projection of μ is a measure of the same type. If for a given one dimensional over $\mathbf{K}$ subspace W in X it is the equality $B(v_q(z),v_q(z)) = 0$ for each $z \in W$, then the projection μ_W of μ is the atomic measure with one atom. Show $B \in L_1(c_0(\omega_0,\mathbf{K}))$ and $\gamma \in c_0(\omega_0,\mathbf{K})$. Let $0 \neq x \in X$ and consider the projection $\pi_x : X \to x\mathbf{K}$. Since $\mu_{x\mathbf{K}}$ is the measure on $Bco(x\mathbf{K})$, then its characteristic functional satisfies Conditions of Theorem 2.30.

Then $\hat{\mu}$ for $x\mathbf{K}$ gives the same characteristic functional of the type

$$\hat{\mu}_{x\mathbf{K}}(z) = s^{[b_x(v^s_q(z))^2]}\chi_{\delta_x}(z)$$

for each $z \in x\mathbf{K}$, where $b_x > 0$ and $\delta_x \in \mathbf{K}$ are constants depending on the parameter $0 \neq x \in X$. Since x and z are arbitrary, then this implies, that $B \in L_1$ and $\gamma \in c_0(\omega_0,\mathbf{K})$.

4. Corollary. *A q-Gaussian measure μ from Proposition* 3 *with $Tr(B) < \infty$ is quasi-invariant and pseudo-differentiable for some $b \in \mathbf{C_s}$ relative to a dense subspace $J_\mu \subset M_\mu = \{x \in X : v^s_q(x) \in E^{1/2}(Y)\}$. Moreover, if B is diagonal, then each one-dimensional projection μ^g has the following characteristic functional:*

$$(i) \quad \hat{\mu}^g(h) = s^{(\sum_j \beta_j |g_j|^q)|h|^q}\chi_{g(\gamma)}(h),$$

where $g = (g_j : j \in \omega_0) \in c_0(\omega_0,\mathbf{K})^*$, $\beta_j > 0$ for each j.

Proof. Using the projective limit reduce consideration to the Banach space X. Take a prime number s such that $s \neq p$ and consider a field $\mathbf{K_s}$ such that $\mathbf{K}$ is compatible with $\mathbf{K_s}$, which is possible, since $\mathbf{K}$ is a finite algebraic extension of $\mathbf{Q_p}$ and it is possible to take in particular $\mathbf{K_s} = \mathbf{Q_s}$. Recall that a group G for which $o(G) \subset o(\mathbf{T_K})$ is called compatible with $\mathbf{K}$, where $o(G)$ denotes the set of all natural numbers for which G has an open subgroup U such that at least one of the elements of the quotient group G/U has order n, $\mathbf{T}$ denotes the group of all roots of 1 and $\mathbf{T_K}$ denotes its subgroup of all elements whose orders are not divisible by the characteristic p of the residue class field k of $\mathbf{K}$. A character of G is a continuous homomorphism $f : G \to \mathbf{T}$. Under point-wise multiplication characters form a group denoted by $G\hat{}$. A group G is called torsional, if each compact subset V of G is contained in a compact subgroup of G.

Theorem 9.14 [Roo78] states: if the field $\mathbf{K}$ is locally compact, then $\mathbf{K}$ is torsional, $\mathbf{K}$ and $\mathbf{K}\hat{}$ are isomorphic as topological groups. Taking any non-constant character ϕ of $\mathbf{K}$ and setting $\phi_s(t) := \phi(st)$ with $s,t \in \mathbf{K}$ one gets an isomorphism $s \mapsto \phi_s$ and a homeomorphism of $\mathbf{K}$ onto $\mathbf{K}\hat{}$.

In view of this theorem $\mathbf{K}\hat{}$ is isomorphic with $\mathbf{K}$. A $\mathbf{K}$-valued character of a group G is a continuous homomorphism $f : G \to \mathbf{T}_{\mathbf{K}}$. The family of all $\mathbf{K}$-valued characters form a group denoted by $G\hat{}_{\mathbf{K}}$. Since $\mathbf{K}$ is compatible with $\mathbf{K_s}$ and $\lim_{n\to\infty} p^n = 0$, then $\mathbf{K}\hat{}$ is isomorphic with $\mathbf{K}\hat{}_{\mathbf{K_s}}$.

If G is a locally compact torsional group compatible with $\mathbf{K}$, then the Fourier-Stieltjes transform of a tight measure $\mu \in M(G)$ is the mapping $\hat{\mu} : G\hat{}_{\mathbf{K}} \to \mathbf{K}$ defined by the formula: $\hat{\mu}(g) := \int_G \chi(x)\mu(dx)$, where $\chi \in G\hat{}_{\mathbf{K}}$. Moreover, the Fourier-Stieltjes transform induces a Banach algebra isomorphism $L(G,\mathcal{R},w,\mathbf{K})$ with $C_\infty(G\hat{}_{\mathbf{K}},\mathbf{K})$, where w is a nontrivial Haar $\mathbf{K}$-valued measure on G. The just above formulated statement is proved in the Schikhof's theorem (see also § 9.21 in [Roo78]).

Therefore, in this situation there exists the Banach algebra isomorphism of $L(\mathbf{K},\mathcal{R},w,\mathbf{K_s})$ with $C_\infty(\mathbf{K}\hat{}_{\mathbf{K_s}},\mathbf{K_s})$.

Therefore, from the proof above and Theorem 3.5 it follows, that the measure $\mu_{q,B,\gamma}$ is quasi-invariant relative to shifts on vectors from the dense subspace X' in X such that $X' = \{x \in X : v_q^s(x) \in E^{1/2}(Y)\}$, which is $\mathbf{K}$-linear, since B is $\mathbf{R}$-bilinear and $B(y,z) =: (Ey,z)$ for each $y,z \in Y$ and $v_q^s(ax) = |a|^{q/2} v_q^s(x)$ and $v_q^s(x_j + t_j) \le \max(v_q^s(x_j), v_q^s(t_j))$ for each $x,t \in X$ and each $a \in \mathbf{K}$, where E is nondegenerate positive definite of trace class $\mathbf{R}$-linear operator on Y, $x = \sum_j x_j e_j$, $x_j \in \mathbf{K}$, since $l_2^* = l_2$ and E can be extended from $\mathsf{D}_{B,Y}$ on Y.

Consider s^{a+ib} as in § 4.1. Mention, that $|(|z|_p)|_s = 1$ for each $z \in \mathbf{K}$, where the field $\mathbf{K}$ is compatible with $\mathbf{K_s}$.

The pseudo-differential operator has the form:

$$PD(b, f(x)) := \int_{\mathbf{K}} [f(x) - f(y)] s^{(-1-b)\times ord_p(x-y)} w(dy),$$

where w is the Haar $\mathbf{K_s}$-valued measure on $Bco(\mathbf{K})$, $b \in \mathbf{C_s}$, particularly, also for $f(x) := \mu(-xz + A)$ for a given $z \in X'$, $A \in Bco(X)$, where $x,y \in \mathbf{K}$. Using the Fourier-Stieltjes transform write it in the form: $PD(b, f(x)) = F_v^{-1}(\xi(v)\psi(v))$, where $\xi(v) := [F_y(f(x) - f(y))](v)$, $\psi(v) := [F_y(s^{(-1-b)\times ord_p(y)})](v)$, F_y means the Fourier-Stieltjes operator by the variable y. Denoting $A - xz =: S$ we can consider $f(x) = 0$ and $f(y) = \mu((x-y)z + S) - \mu(S)$, since $S \in Bco(X)$. Then $f(y) = \int_S (\mu((x-y)+dg) - \mu(dg)) = \int_S [\rho_\mu(y-x,g) - 1]\mu(dg)$. The constant function $h(g) = 1$ is evidently pseudo-differentiable of order b for each $b \in \mathbf{C_s}$. Hence the pseudo-differentiability of μ of order b follows from the existence of pseudo-differential of the quasi-invariance factor $\rho_\mu(y, g + x)$ of order b for μ-almost every $g \in X$.

In view of Theorem 3.5 and the Fourier-Stieltjes operator isomorphism of Banach algebras $L(\mathbf{K},\mathcal{R},w,\mathbf{K_s})$ and $C_\infty(\mathbf{K}\hat{}_{\mathbf{K_s}},\mathbf{K_s})$ the pseudo-differentiability of ρ_μ follows from the existence of $F^{-1}(\hat{\mu}\psi)$, where $\hat{\mu}$ is the characteristic functional of μ. We have

$$(ii) \quad F(f)(y) = \int_{\mathbf{K}} \chi(xy) f(x) w(dx)$$

$$= \int_{\mathbf{K}} \chi(z) f(z/y)[|y|_p]^{-1} w(dz)$$

for each $y \neq 0$, where $x, y, z \in \mathbf{K}$, particularly, for $f(x) = s^{-(1+b)\times ord_p(x)}$ we have $f(z/y) = f(z)f(-y)$ and $F(f)(y) = \Gamma^{\mathbf{K},s}(1+b)f(-y)|y|_p^{-1}$, where

$$(iii) \quad \Gamma^{\mathbf{K},s}(b) := \int_{\mathbf{K}} \chi(z) s^{-b\times ord_p(x)} w(dz),$$

$f(-y) = s^{(1+b)\times ord_p(y)}$, since $ord_p(z/y) = ord_p(x) - ord_p(y)$.

For a nontrivial character of an order $m \in \mathbf{Z}$ from the definition it follows, that $\Gamma^{\mathbf{K},s}(b) \neq 0$ for each b with $Re(b) \neq 0$, since $|s^{-bn}|_s = s^{Re(b)n}$ for each $n \in \mathbf{Z}$. Therefore, $\psi(y) = s^{(1+b)\times ord_p(y)}|y|_p^{-1}$, consequently, $|\psi(y)|_s = s^{-(1+Re(b))\times ord_p(y))}$ for each $y \neq 0$, since $|(|y|_p)|_s = 1$. On the other hand, $|\hat{\mu}(z)| = s^{-B(v_q^s(z), v_q^s(z))}$ and $F^{-1}(\hat{\mu}\psi)$ exists for each $b \in \mathbf{C_s}$ with $Re(b) > -1$, since $Tr(B) < \infty$, which is correct, since $\mathbf{C_s}$ is algebraically isomorphic with $\mathbf{C}$ and $\Gamma_{\mathbf{U_s}} \supset (0, \infty)$.

5. Corollary. *Let X be a complete locally $\mathbf{K}$-convex space of separable type over a local field $\mathbf{K}$, then for each constant $q > 0$ there exists a nondegenerate symmetric positive definite operator $B \in L_1$ such that a q-Gaussian quasi-measure is a measure on $Bco(X)$ and each its one dimensional over $\mathbf{K}$ projection is absolutely continuous relative to the nonnegative Haar measure on $\mathbf{K}$.*

Proof. It is analogous to that of Corollary I.6.8. For each $\mathbf{K}$-linear finite dimensional over $\mathbf{K}$ subspace S a projection μ^S of μ on $S \subset X$ exists and its density $\mu^S(dx)/w(dx)$ relative to the nondegenerate $\mathbf{K_s}$-valued Haar measure w on S is the inverse Fourier-Stieltjes transform $F^{-1}(\hat{\mu}|_{S^*})$ of the restriction of $\hat{\mu}$ on S^*. For $B \in L_1$ each one dimensional projection of μ corresponding to $\hat{\mu}$ has a density that is a continuous function belonging to $L(\mathbf{K}, Bco(\mathbf{K}), w, \mathbf{K_s})$.

6. Proposition. *Let $\mu_{q,B,\gamma}$ and $\mu_{q,E,\delta}$ be two q-Gaussian measures with correlation operators B and E of class L_1, then there exists a convolution of these measures $\mu_{q,B,\gamma} * \mu_{q,E,\delta}$, which is a q-Gaussian measure $\mu_{q,B+E,\gamma+\delta}$.*

Proof. It is analogous to that of I.6.9 with the substitution of $Bf(X)$ on $Bco(X)$.

6.1. Remark and Definition. A measurable space (Ω, F) with a probability $\mathbf{K_s}$-valued measure λ on a covering ring F of a set Ω is called a probability space and it is denoted by $(\Omega, \mathsf{F}, \lambda)$. Points $\omega \in \Omega$ are called elementary events and values $\lambda(S)$ probabilities of events $S \in \mathsf{F}$. A measurable map $\xi : (\Omega, \mathsf{F}) \to (X, \mathsf{B})$ is called a random variable with values in X, where B is a covering ring such that $\mathsf{B} \subset Bco(X)$, $Bco(X)$ is the ring of all clopen subsets of a locally $\mathbf{K}$-convex space X, $\xi^{-1}(\mathsf{B}) \subset \mathsf{F}$, where $\mathbf{K}$ is a non-Archimedean field complete as an ultra-metric space.

The random variable ξ induces a normalized measure $\nu_\xi(A) := \lambda(\xi^{-1}(A))$ in X and a new probability space (X, B, ν_ξ).

Let T be a set with a covering ring $\mathcal{R}$ and a measure $\eta : \mathcal{R} \to \mathbf{K_s}$. Consider the following Banach space $L^q(T, \mathcal{R}, \eta, H)$ as the completion of the set of all $\mathcal{R}$-step functions $f : T \to H$ relative to the following norm:

(1) $\|f\|_{\eta,q} := \sup_{t\in T} \|f(t)\|_H N_\eta(t)^{1/q}$ for $1 \le q < \infty$ and

(2) $\|f\|_{\eta,\infty} := \sup_{1\le q<\infty} \|f(t)\|_{\eta,q}$, where H is a Banach space over $\mathbf{K}$. For $0 < q < 1$

this is the metric space with the metric

$$\rho_q(f,g) := \sup_{t\in T} \|f(t)-g(t)\|_H N_\eta(t)^{1/q}. \tag{3}$$

If H is a complete locally **K**-convex space, then H is a projective limit of Banach spaces $H = \lim\{H_\alpha, \pi^\alpha_\beta, \Upsilon\}$, where Υ is a directed set, $\pi^\alpha_\beta : H_\alpha \to H_\beta$ is a **K**-linear continuous mapping for each $\alpha \ge \beta$, $\pi_\alpha : H \to H_\alpha$ is a **K**-linear continuous mapping such that $\pi^\alpha_\beta \circ \pi_\alpha = \pi_\beta$ for each $\alpha \ge \beta$ (see also § 6.205 [NB85]). Each norm p_α on H_α induces a pre-norm $\tilde{p}_\alpha$ on H. If $f : T \to H$, then $\pi_\alpha \circ f =: f_\alpha : T \to H_\alpha$. In this case $L^q(T, \mathcal{R}, \eta, H)$ is defined as a completion of a family of all step functions $f : T \to H$ relative to the family of pre-norms

(1′) $\|f\|_{\eta,q,\alpha} := \sup_{t\in T} \tilde{p}_\alpha(f(t))N_\eta(t)^{1/q}$, $\alpha \in \Upsilon$, for $1 \le q < \infty$ and

(2′) $\|f\|_{\eta,\infty,\alpha} := \sup_{1\le q<\infty} \|f(t)\|_{\eta,q,\alpha}$, $\alpha \in \Upsilon$, or pseudo-metrics

(3′) $\rho_{q,\alpha}(f,g) := \sup_{t\in T} \tilde{p}_\alpha(f(t)-g(t))N_\eta(t)^{1/q}$, $\alpha \in \Upsilon$, for $0 < q < 1$. Therefore, $L^q(T, \mathcal{R}, \eta, H)$ is isomorphic with the projective limit
$\lim\{L^q(T, \mathcal{R}, \eta, H_\alpha), \pi^\alpha_\beta, \Upsilon\}$.

For $q = 1$ we write simply $L(T, \mathcal{R}, \eta, H)$ and $\|f\|_\eta$. This definition is correct, since $\lim_{q\to\infty} a^{1/q} = 1$ for each $\infty > a > 0$. For example, T may be a subset of **R**. Let $\mathbf{R_d}$ be the field **R** supplied with the discrete topology. Since the cardinality $card(\mathbf{R}) = \mathsf{c} = 2^{\aleph_0}$, then there are bijective mappings of **R** on $Y_1 := \{0,\dots,b\}^{\mathbf{N}}$ and also on $Y_2 := \mathbf{N}^{\mathbf{N}}$, where b is a positive integer number. Supply $\{0,\dots,b\}$ and **N** with the discrete topologies and Y_1 and Y_2 with the product topologies.

Then zero-dimensional spaces Y_1 and Y_2 supply with covering separating rings $\mathcal{R}_1$ and $\mathcal{R}_2$ contained in $Bco(Y_1)$ and $Bco(Y_2)$ respectively. Certainly such separating covering ring in **R** induced from Y_1 or Y_2 is not related with the standard (Euclidean) metric in **R**. Therefore, for the space $L^q(T, \mathcal{R}, \eta, H)$ we can consider $t \in T$ as the real time parameter. If $T \subset \mathbf{F}$ with a non-Archimedean field **F**, then we can consider the non-Archimedean time parameter.

If T is a zero-dimensional T_1-space, then denote by $C^0_b(T,H)$ the Banach space of all continuous bounded functions $f : T \to H$ supplied with the norm:

(4) $\|f\|_{C^0} := \sup_{t\in T} \|f(t)\|_H < \infty$.

If T is compact, then $C^0_b(T,H)$ is isomorphic with the space $C^0(T,H)$ of all continuous functions $f : T \to H$.

For a set T and a complete locally **K**-convex space H over **K** consider the product **K**-convex space $H^T := \prod_{t\in T} H_t$ in the product topology, where $H_t := H$ for each $t \in T$.

Then take on either $X := X(T,H) = L^q(T, \mathcal{R}, \eta, H)$ or $X := X(T,H) = C^0_b(T,H)$ or on $X = X(T,H) = H^T$ a covering ring B such that $\mathsf{B} \subset Bco(X)$. Consider a random variable $\xi : \omega \mapsto \xi(t,\omega)$ with values in (X, B), where $t \in T$.

Events $S_1,\dots,S_n$ are called independent in total if $P(\prod_{k=1}^n S_k) = \prod_{k=1}^n P(S_k)$. Sub-rings $\mathsf{F}_k \subset \mathsf{F}$ are said to be independent if all collections of events $S_k \in \mathsf{F}_k$ are independent in total, where $k = 1,\dots,n$, $n \in \mathbf{N}$. To each collection of random variables ξ_γ on (Ω, F) with $\gamma \in \Upsilon$ is related the minimal ring $\mathsf{F}_\Upsilon \subset \mathsf{F}$ with respect to which all ξ_γ are measurable, where Υ is a set. Collections $\{\xi_\gamma : \gamma \in \Upsilon_j\}$ are called independent if such are F_{Υ_j}, where $\Upsilon_j \subset \Upsilon$ for each $j = 1,\dots,n$, $n \in \mathbf{N}$.

Consider T such that $card(T) > n$. For $X = C^0_b(T,H)$ or $X = H^T$ define $X(T,H;(t_1,\dots,t_n);(z_1,\dots,z_n))$ as a closed sub-manifold in X of all $f : T \to H$, $f \in X$

such that $f(t_1) = z_1, \dots, f(t_n) = z_n$, where $t_1, \dots, t_n$ are pairwise distinct points in T and $z_1, \dots, z_n$ are points in H. For $X = L^q(T, \mathcal{R}, \eta, H)$ and pairwise distinct points $t_1, \dots, t_n$ in T with $N_\eta(t_1) > 0, \dots, N_\eta(t_n) > 0$ define $X(T,H;(t_1,\dots,t_n);(z_1,\dots,z_n))$ as a closed sub-manifold which is the completion relative to the norm $\|f\|_{\eta,q}$ of a family of $\mathcal{R}$-step functions $f: T \to H$ such that $f(t_1) = z_1, \dots, f(t_n) = z_n$. In these cases $X(T,H;(t_1,\dots,t_n);(0,\dots,0))$ is the proper **K**-linear subspace of $X(T,H)$ such that $X(T,H)$ is isomorphic with $X(T,H;(t_1,\dots,t_n);(0,\dots,0)) \oplus H^n$, since if $f \in X$, then $f(t) - f(t_1) =: g(t) \in X(T,H;t_1;0)$ (in the third case we use that $T \in \mathcal{R}$ and hence there exists the embedding $H \hookrightarrow X$). For $n = 1$ and $t_0 \in T$ and $z_1 = 0$ we denote $X_0 := X_0(T,H) := X(T,H;t_0;0)$.

6.2. Definitions. We define a (non-Archimedean) stochastic process $w(t,\omega)$ with values in H as a random variable such that:

(i) the differences $w(t_4,\omega) - w(t_3,\omega)$ and $w(t_2,\omega) - w(t_1,\omega)$ are independent for each chosen (t_1,t_2) and (t_3,t_4) with $t_1 \ne t_2$, $t_3 \ne t_4$, such that either t_1 or t_2 is not in the two-element set $\{t_3,t_4\}$, where $\omega \in \Omega$;

(ii) the random variable $\omega(t,\omega) - \omega(u,\omega)$ has a distribution $\mu^{F_{t,u}}$, where μ is a probability $\mathbf{K_s}$-valued measure on $(X(T,H),\mathsf{B})$ from § 6.1, $\mu^g(A) := \mu(g^{-1}(A))$ for $g: X \to H$ such that $g^{-1}(\mathcal{R}_H) \subset \mathsf{B}$ and each $A \in \mathcal{R}_H$, a continuous linear operator $F_{t,u}: X \to H$ is given by the formula $F_{t,u}(w) := w(t,\omega) - w(u,\omega)$ for each $w \in L^q(\Omega,\mathsf{F},\lambda;X)$, where $1 \le q \le \infty$, $\mathcal{R}_H$ is a covering ring of H such that $F_{t,u}^{-1}(\mathcal{R}_H) \subset \mathsf{B}$ for each $t \ne u$ in T;

(iii) we also put $w(0,\omega) = 0$, that is, we consider a **K**-linear subspace $L^q(\Omega,\mathsf{F},\lambda;X_0)$ of $L^q(\Omega,\mathsf{F},\lambda;X)$, where $\Omega \ne \emptyset$, X_0 is the closed subspace of X as in § 6.1.

7. Definition. Let B and q be as in § 2 and denote by $\mu_{q,B,\gamma}$ the corresponding q-Gaussian $\mathbf{K_s}$-valued measure on H. Let ξ be a stochastic process with a real time $t \in T \subset \mathbf{R}$ (see Definition 6.2), then it is called a non-Archimedean q-Wiener process with real time (and controlled by $\mathbf{K_s}$-valued measure), if

$(ii)'$ the random variable $\xi(t,\omega) - \xi(u,\omega)$ has a distribution $\mu_{q,(t-u)B,\gamma}$ for each $t \ne u \in T$.

Let ξ be a stochastic process with a non-Archimedean time $t \in T \subset \mathbf{F}$, where $\mathbf{F}$ is a local field, then ξ is called a non-Archimedean q-Wiener process with $\mathbf{F}$-time (and controlled by $\mathbf{K_s}$-valued measure), if

(ii)" the random variable $\xi(t,\omega) - \xi(u,\omega)$ has a distribution $\mu_{q,ln[\chi_{\mathbf{F}}(t-u)]B,\gamma}$ for each $t \ne u \in T$, where $\chi_{\mathbf{F}}: \mathbf{F} \to \mathbf{T}$ is a continuous character of $\mathbf{F}$ as the additive group (see §2.5).

8. Proposition. *For each given q-Gaussian measure a non-Archimedean q-Wiener process with real (* **F** *respectively) time exists.*

Proof. In view of Proposition 6 for each $t > u > b$ a random variable $\xi(t,\omega) - \xi(b,\omega)$ has a distribution $\mu_{q,(t-b)B,\gamma}$ for real time parameter. If t, u, b are pairwise different points in $\mathbf{F}$, then $\xi(t,\omega) - \xi(b,\omega)$ has a distribution $\mu_{q,ln[\chi_{\mathbf{F}}(t-b)]B,\gamma}$, since $ln[\chi_{\mathbf{F}}(t-u)] + ln[\chi_{\mathbf{F}}(u-b)] = ln[\chi_{\mathbf{F}}(t-b)]$. This induces the Markov quasi-measure $\mu^{(q)}_{x_0,\tau}$ on $(\prod_{t\in T}(H_t,\mathsf{U}_t))$, where $H_t = H$ and $\mathsf{U}_t = Bco(H)$ for each $t \in T$. In view of Theorem 2.39 there exists an abstract probability space $(\Omega,\mathsf{F},\lambda)$, consequently, the corresponding space $L(\Omega,\mathsf{F},\lambda,\mathbf{K_s})$ exists.

9. Proposition. *Let ξ be a q-Gaussian process with values in a Banach space $H = c_0(\alpha,\mathbf{K})$ a time parameter $t \in T$ (controlled by a $\mathbf{K_s}$-valued measure) and a positive definite correlation operator B of trace class and $\gamma = 0$, where $card(\alpha) \le \aleph_0$, either $T \subset \mathbf{R}$ or $T \subset \mathbf{F}$. Then either*

$$(i) \quad \lim_{N\in\alpha} M_t[v_q^s(e^1(\xi(t,\omega))^2 + \dots + v_q^s(e^N(\xi(t,\omega)))^2] = tTr(B) \text{ or}$$

(ii) $\lim_{N\in\alpha} M_t[v_q^s(e^1(\xi(t,\omega))^2+\cdots+v_q^s(e^N(\xi(t,\omega))^2]=[ln(\chi_{\mathbf{F}}(t))]Tr(B)$ *respectively.*

Proof. Define $\mathbf{U_s}$-valued moments

$$m_k^q(e^{j_1},\dots,e^{j_k}) := \int_H v_{2q}^s(e^{j_1}(x))\cdots v_{2q}^s(e^{j_k}(x))\mu_{q,B,\gamma}(dx)$$

for linear continuous functionals $e^{j_1},\dots,e^{j_k}$ on H such that $e^l(e_j)=\delta_j^l$, where $\{e_j : j\in\alpha\}$ is the standard orthonormal base in H.

Consider the operator

(iii) ${}_P\partial^u\psi(x) := F^{-1}(\hat{f}_{u-1}(y)\hat{\psi}(y)|y|_p)(x)$,

where $f_u(x) := s^{-(1+u)\times ord_p(x)}/\Gamma^{\mathbf{K},s}(1+u)$ and $F(f_u)(y)=\Gamma^{\mathbf{K},s}(1+u)f_u(-y)|y|_p^{-1}$ (see §4), where F denotes the Fourier-Stieltjes operator defined with the help of the $\mathbf{K_s}$-valued Haar measure w on $Bco(\mathbf{K})$, $F(\psi) =: \hat{\psi}$, $Re(u)\neq -1$, $\psi : \mathbf{K}\to\mathbf{K_s}$. Then

(iv) ${}_P\partial^u f_b(x) = F^{-1}(\Gamma^{\mathbf{K},s}(u)f_{u-1}(-y)\Gamma^{\mathbf{K},s}(1+b)f_b(-y)|y|_p^{-1}) = f_{(u+b)}(x)$

for each u with $Re(u)\neq 0$, since

$F^{-1}(s^{-(1+u+b)\times ord_p(-y)}|y|_p^{-1})(x) = (\Gamma^{\mathbf{K},s}(1+u+b))^{-1}s^{-(1+u+b)\times ord_p(-y)}(x)$.

For $u=1$ we write shortly ${}_P\partial^1 = {}_P\partial$ and ${}_P\partial_j^u$ means the operator of partial pseudo-differential (with weight multiplier) given by Equation (iii) by the variable x_j. A function ψ for which ${}_P\partial_j^u\psi$ exists is called pseudo-differentiable (with weight multiplier) of order u by variable x_j. Then

$$m_{2k}^{q/2}(e^{j_1},\dots,e^{j_{2k}})(\Gamma^{\mathbf{K},s}(q/2))^{2k} := \int_H s^{-q\ ord_p(x_{j_1})/2}\cdots s^{-q\ ord_p(x_{j_{2k}})/2}\mu_{q,B,\gamma}(dx)$$

$$= {}_P\partial_{j_1}^{q/2}\cdots {}_P\partial_{j_{2k}}^{q/2}\hat{\mu}_{q,B,\gamma}(0) = ([{}_PD^{q/2}]^{2k}\hat{\mu}(x))|_{x=0}.(e^{j_1},\dots,e^{j_{2k}}),$$

where $({}_PD^{q/2}f(x)).e^j := {}_P\partial_j f(x)$. Therefore,

(v) $m_{2k}^{q/2}(e^{j_1},\dots,e^{j_{2k}})(\Gamma^{\mathbf{K},s}(q/2))^{2k}$
$= (k!)^{-1}[{}_PD^{q/2}]^{2k}[B(v_q^s(z),v_q^s(z)]^k.(e_{j_1},\dots,e_{j_{2k}})$
$= (k!)^{-1}\sum_{\sigma\in\Sigma_{2k}} B_{\sigma(j_1),\sigma(j_2)}\cdots B_{\sigma(j_{2k-1}),\sigma(j_{2k})}$,

since $\gamma=0$ and $\chi_\gamma(z)=1$, where Σ_k is the symmetric group of all bijective mappings σ of the set $\{1,\dots,k\}$ onto itself, $B_{l,j} := B(e_j,e_l)$, since $Y^*=Y$ for $Y=l_2(\alpha,\mathbf{R})$. Therefore, for each $B\in L_1$ and $A\in L_\infty$ we have $\int_H A(v_q(x),v_q(x))\mu_{q,B,0}(dx) = \lim_{N\in\alpha}\sum_{j=1}^N\sum_{k=1}^N A_{j,k}m_2^{q/2}(e_j,e_k) = Tr(AB)$, since $\mathbf{C_s}\subset\mathbf{U_s}$ and algebraically $\mathbf{C_s}$ is isomorphic with $\mathbf{C}$.

In particular for $A=I$ and $\mu_{q,tB,0}$ corresponding to the transition measure of $\xi(t,\omega)$ we get Formula (i) for a real time parameter, using $\mu_{q,ln[\chi_{\mathbf{F}}(t)]B,0}$ we get Formula (ii) for a time parameter belonging to $\mathbf{F}$, since $\xi(t_0,\omega)=0$ for each ω.

10. Corollary. *Let* $H=\mathbf{K}$ *and* ξ, $B=1$, γ *be as in Proposition* 9, *then*

$$(i)\quad M\left(\int_{t\in[a,b]}\phi(t,\omega)v_{2q}^s(d\xi(t,\omega))\right) = M\left[\int_a^b\phi(t,\omega)dt\right]$$

for each $a<b\in T$ *with real time, where* $\phi(t,\omega)\in L(\Omega,\mathsf{U},\lambda,C_0^0(T,\mathbf{R}))$ $\xi\in L(\Omega,\mathsf{U},\lambda,X_0(T,\mathbf{K}))$, $(\Omega,\mathsf{U},\lambda)$ *is a probability measure space.*

Proof. Since

$$\int_{t\in[a,b]}\phi(t,\omega)v^s_{2q}(d\xi(t,\omega))=\lim_{\max_j(t_{j+1}-t_j)\to 0}\sum_{j=1}^{N}\phi(t_j,\omega)v^s_q(\xi(t_{j+1},\omega)-\xi(t_j,\omega))$$

for λ-almost all $\omega\in\Omega$, since $\mathbf{C_s}\subset\mathbf{U_s}$ and $\mathbf{C_s}$ is algebraically isomorphic with $\mathbf{C}$, then from the application of Formula 9.(i) to each $v^s_{2q}(\xi(t_{j+1},\omega)-\xi(t_j,\omega))$ and the existence of the limit by finite partitions $a=t_1<t_2<\cdots<t_{N+1}=b$ of the segment $[a,b]$ it follows Formula 10.(i).

11. Definitions and Notes. Consider a pseudo-differential operator on $H=c_0(\alpha,\mathbf{K})$ such that

$$(i)\quad \mathsf{A}=\sum_{0\le k\in\mathbf{Z};j_1,\dots,j_k\in\alpha}(-i)^k b^k_{j_1,\dots,j_k}\, {}_P\partial_{j_1}\cdots {}_P\partial_{j_k},$$

where $b^k_{j_1,\dots,j_k}\in\mathbf{R}$, ${}_P\partial_{j_k}:={}_P\partial^1_{j_k}$. If there exists $n:=\max\{k: b^k_{j_1,\dots,j_k}\ne 0, j_1,\dots,j_k\in\alpha\}$, then n is called an order of A, $Ord(\mathsf{A})$, where ${}_P\partial_j$ is defined by Formula 9.(iii). If $\mathsf{A}=0$, then by definition $Ord(\mathsf{A})=0$. If there is not any such finite n, then $Ord(\mathsf{A})=\infty$. We suppose that the corresponding form $\tilde{A}$ on $\bigoplus_k Y^k$ is continuous into $\mathbf{C}$, where

$$(ii)\quad \tilde{A}(y)=-\sum_{0\le k\in\mathbf{Z};j_1,\dots,j_k\in\alpha}(-i)^k b^k_{j_1,\dots,j_k} y_{j_1}\cdots y_{j_k}/lns,$$

$y\in l_2(\alpha,\mathbf{R})=:Y$. If $\tilde{A}(y)>0$ for each $y\ne 0$ in Y, then A is called strictly elliptic pseudo-differential operator.

Let X be a complete locally $\mathbf{K}$-convex space, let Z be a complete locally $\mathbf{U_s}$-convex space. For $0\le n\in\mathbf{R}$ a space of all functions $f:X\to Z$ such that $f(x)$ and $({}_PD^kf(x)).(y^1,\dots,y^{l(k)})$ are continuous functions on X for each $y^1,\dots,y^{l(k)}\in\{e^1,e^2,e^3,\dots\}$, $l(k):=[k]+sign\{k\}$ for each $k\in\mathbf{N}$ such that $k\le[n]$ and also for $k=n$ is denoted by ${}_P\mathcal{C}^n(X,Z)$ and $f\in{}_P\mathcal{C}^n(X,Z)$ is called n times continuously pseudo-differentiable, where $[n]\le n$ is an integer part of n, $1>\{n\}:=n-[n]\ge 0$ is a fractional part of n. Then ${}_P\mathcal{C}^\infty(X,Z):=\bigcap_{n=1}^\infty {}_P\mathcal{C}^n(X,Z)$ denotes a space of all infinitely pseudo-differentiable functions.

Embed $\mathbf{R}$ into $\mathbf{C_s}$ and consider the function $v^s_2:\mathbf{U_p}\to\mathbf{R}\subset\mathbf{C_s}$, then for $t=v^s_2(\theta)$, $\theta\in\mathbf{K}\subset\mathbf{U_p}$, put $\partial_t u(t,x):=\lim_{\theta,\mathbf{K},\theta\in\mathbf{K},v^s_2(\theta)\to t}{}_P\partial_\theta u(v^s_2(\theta),x)$ for $t\ge 0$, when it exists by the filter of local subfields $\mathbf{K}\subset\mathbf{C_p}$, which is correct, since $v^s_2(\mathbf{U_p})=[0,\infty)$, $\bigcup_{\mathbf{K}\subset\mathbf{C_p}}\mathbf{K}$ is dense in $\mathbf{C_p}$, $\Gamma_{\mathbf{C_p}}=(0,\infty)\cap\mathbf{Q}$.

12. Theorem. *Let* A *be a strictly elliptic pseudo-differential operator on* $H=c_0(\alpha,\mathbf{K})$, $card(\alpha)\le\aleph_0$, *and let* $t\in T=[0,b]\subset\mathbf{R}$. *Suppose also that* $u_0(x-y)\in L(H,Bco(H),\mu_{t\tilde{A}},\mathbf{U_s})$ *for each marked* $y\in H$ *as a function by* $x\in H$, $u_0(x)\in{}_P\mathcal{C}^{Ord(\mathsf{A})}(H,\mathbf{U_s})$. *Then the non-Archimedean analog of the Cauchy problem*

$$(i)\quad \partial_t u(t,x)=\mathsf{A}u,\quad u(0,x)=u_0(x)$$

has a solution given by

$$(ii)\quad u(t,x)=\int_H u_0(x-y)\mu_{t\tilde{A}}(dy),$$

where $\mu_{t\tilde{A}}$ *is a* $\mathbf{K_s}$*-valued measure on* H *with a characteristic functional* $\hat{\mu}_{t\tilde{A}}(z):=s^{t\tilde{A}(v^s_2(z))}$.

Proof. In accordance with § § 2 and 11 we have $Y = l_2(\alpha, \mathbf{R})$. The function $s^{t\tilde{A}(v_2^s(z))}$ is continuous on $H \hookrightarrow H^*$ for each $t \in \mathbf{R}$ such that the family H of continuous $\mathbf{K}$-linear functionals on H separates points in H. In view of Theorem 2.30 above it defines a tight measure on H for each $t > 0$. The functional $\tilde{A}$ on each ball of radius $0 < R < \infty$ in Y is a uniform limit of its restrictions $\tilde{A}|_{\bigoplus_k [span_{\mathbf{K}}(e_1,\dots,e_n)]^k}$, when n tends to the infinity, since $\tilde{A}$ is continuous on $\bigoplus_k Y^k$. Since $u_0(x-y) \in L(H, Bco(H), \mu_{t\tilde{A}}, \mathbf{U_s})$ and a space of cylindrical functions is dense in the latter Banach space over $\mathbf{U_s}$, then in view of theorems about the isomorphism of $\mathbf{K}$ with $\mathbf{K}\hat{}$ for a locally compact field and the Schikhof's theorem about the isomorphism $L(G) \simeq C_\infty(G\hat{}_{\mathbf{K}})$ formulated above (see also Theorems 9.14 and 9.21 in [Roo78]) and the Fubini theorem it follows that $\lim_{P \to I} \mathsf{F}_{Px} u_0(Px))\hat{\mu}_{t\tilde{A}}(y+Px)$ converges in $L(H, Bco(H), \mu_{t\tilde{A}}, \mathbf{U_s})$ for each t, since $\mu_{t_1\tilde{A}} * \mu_{t_2\tilde{A}} = \mu_{(t_1+t_2)\tilde{A}}$ for each t_1, t_2 and $t_1 + t_2 \in T$, where P is a projection on a finite dimensional over $\mathbf{K}$ subspace $H_P := P(H)$ in H, $H_P \hookrightarrow H$, P tends to the unit operator I in the strong operator topology, $F_{Px}u_0(Px)$ denotes a Fourier transform by the variable $Px \in H_P$.

We consider now the function $v := F_x(u)$, then $\partial_t v(t,x) = -\tilde{A}(v_2^s(x))v(t,x) lns$, consequently, $v(t,x) = v_0(x)s^{t\tilde{A}(v_2^s(x))}$. From $u(t,x) = F_x^{-1}(v(t,x))$, where $F_x(u(t,x)) = \lim_{n\to\infty} F_{x_1,\dots,x_n} u(t,x)$. Therefore, $u(t,x) = u_0(x) * [F_x^{-1}(\hat{\mu}_{t\tilde{A}})] = \int_H u_0(x-y)\mu_{t\tilde{A}}(dy)$, since $u_0(x-y) \in L(H, Bco(H), \mu_{t\tilde{A}}, \mathbf{U_s})$ and $\mu_{t\tilde{A}}$ is the tight measure on $Bco(H)$.

14. Note. In the particular case of $Ord(\mathsf{A}) = 2$ and $\tilde{A}$ corresponding to the Laplace operator, that is, $\tilde{A}(y) = \sum_{l,j} g_{l,j} y_l y_j$, Equation 12.$(i)$ is (the non-Archimedean analog of) the heat equation on H.

For $Ord(\mathsf{A}) < \infty$ the form $\tilde{A}_0(y)$ corresponding to sum of terms with $k = Ord(\mathsf{A})$ in Formula 11.(ii) is called the principal symbol of operator A. If $\tilde{A}_0(y) > 0$ for each $y \neq 0$, then A is called an elliptic pseudo-differential operator. Evidently, Theorem 13 is true for elliptic A of $Ord(\mathsf{A}) < \infty$.

15. Remark and Definitions. Let linear spaces X over $\mathbf{K}$ and Y over $\mathbf{R}$ be as in § 4 and B be a symmetric nonnegative definite (bilinear) operator on a dense $\mathbf{R}$-linear subspace $\mathsf{D}_{B,Y}$ in Y^*. A quasi-measure μ with a characteristic functional

$$\hat{\mu}(\zeta, x) := s^{\zeta B(v_q^s(z), v_q^s(z))}\chi_\gamma(z)$$

for a parameter $\zeta \in \mathbf{C_s}$ with $Re(\zeta) \geq 0$ defined on $\mathsf{D}_{q,B,X}$ we call an $\mathbf{U_s}$-valued (non-Archimedean analog of Feynman) quasi-measure and we denote it by $\mu_{q,\zeta B,\gamma}$ also, where $\mathsf{D}_{q,B,X} := \{z \in X^* :$ there exists $j \in \Upsilon$ such that $z(x) = z_j(\phi_j(x))\ \forall x \in X,\ v_q^s(z) \in \mathsf{D}_{B,Y}\}$.

16. Proposition. *Let $X = \mathsf{D}_{q,B,X}$ and B be positive definite, then for each function $f(z) := \int_X \chi_z(x)\nu(dx)$ with an $\mathbf{U_s}$-valued tight measure ν of finite norm and each $Re(\zeta) > 0$ there exists*

$$(i) \quad \int_X f(z)\mu_{\zeta B}(dz) = \lim_{P\to I}\int_X f(Pz)\mu_{\zeta B}^{(P)}(dz)$$

$$= \int_X s^{(\zeta B(v_q(z), v_q(z)))}\chi_\gamma(z)\nu(dz),$$

where $\mu^{(P)}(P^{-1}(A)) := \mu(P^{-1}(A))$ for each $A \in Bco(X_P)$, $P : X \to X_P$ is a projection on a $\mathbf{K}$-linear subspace X_P, a convergence $P \to I$ is considered relative to a strong operator topology.

Proof. From the use of the projective limit decomposition of X and Theorem 2.37 it follows, that there exists

$$(ii) \quad \int_X f(z)\mu_{\zeta B}(dz) = \lim_{P\to I}\int_X f(Pz)\mu^{(P)}_{\zeta B}(dz).$$

Then for each finite dimensional over $\mathbf{K}$ subspace X_P

$$(iii) \quad \int_X f(Pz)\mu^{(P)}_{\zeta B}(dz) = \int_{X_P}\{s^{\zeta B(v^s_q(z),v^s_q(z)))}\chi_\gamma(z)\}|_{X_P}\nu^{X_P}(dz),$$

since ν is tight and hence each ν^{X_P} is tight. Each measure ν_j is tight, then due to Lemma 2.3 and § 2.5 above there exists the limit

$$\lim_{P\to I}\int_{X_P}\{s^{\zeta B(v^s_q(z),v^s_q(z))}\chi_\gamma(z)\}|_{X_P}\nu^{X_P}(dz)$$

$$= \int_X s^{\zeta B(v^s_q(z),v^s_q(z)))}\chi_\gamma(z)\nu(dz).$$

17. Proposition. *If conditions of Proposition* 16 *are satisfied and*

$$(i) \quad f(Px)\in L(X_P, Bco(w^{X_P}),\mathbf{U_s})$$

for each finite dimensional over $\mathbf{K}$ *subspace* X_P *in* X *and*

$$(ii) \quad \lim_{R\to\infty}\sup_{|x|\le R}|f(x)| = 0,$$

then Formula 16(i) *is accomplished for* ζ *with* $Re(\zeta)=0$*, where* w^{X_P} *is a nondegenerate* $\mathbf{K_s}$*-valued Haar measure on* X_P*.*

Proof. In view of Theorem 2.37 for the consistent family of measures $\{f(Px)\mu^{X_P}_{q,iB,\gamma}(dPx) : P\}$ (see § 2.36) there exists a measure on $(X,\mathcal{R})$, where projection operators P are associated with a chosen basis in X. The finite dimensional over $\mathbf{K}$ distribution $\mu^{X_P}_{q,iB,\gamma}/w^{X_P}(dx) = F^{-1}(\hat{\mu}_{q,iB,\gamma})|_{X_P})$ is in $C_\infty(X_P,\mathbf{U_s})$ due to Theorem 9.21 [Roo78], since $\hat{\mu}\in L(X_P, Bco(X_p), w^{X_P},\mathbf{U_s})$. In view of Condition 17.(i,ii) above and the Fubini theorem and using the Fourier-Stieltjes transform we get Formulas 16(ii,iii). From the taking the limit by $P\to I$ Formula 16.(i) follows. This means that $\mu_{q,\zeta B,\gamma}$ exists in the sense of distributions.

18. Remark. Put

$$(i) \quad {}_F\int_X f(x)\mu_{q,iB,\gamma}(dx) := \lim_{\zeta\to i}\int_X f(x)\mu_{q,\zeta B,\gamma}(dx)$$

if such limit exists. If conditions of Proposition 17 are satisfied, then $\psi(\zeta) := \int_X f(x)\mu_{q,\zeta B,\gamma}(dx)$ is the pseudo-differentiable of order 1 function by ζ on the set $\{\zeta\in\mathbf{C_s} : Re(\zeta)>0\}$ and it is continuous on the subset $\{\zeta\in\mathbf{C_s} : Re(\zeta)\ge 0\}$, consequently,

$$(ii) \quad {}_F\int_X f(x)\mu_{q,iB,\gamma}(dx) = \int_X s^{\{iB(v^s_q(x),v^s_q(x))\}}\chi_\gamma(x)\nu(dx).$$

Above non-Archimedean analogs of Gaussian measures with specific properties were defined. Nevertheless, there do not exist usual Gaussian $\mathbf{K_s}$-valued measures on non-Archimedean Banach spaces.

19. Theorem. *Let X be a Banach space of separable type over a locally compact non-Archimedean field $\mathbf{K}$. Then on $Bco(X)$ there does not exist a nontrivial $\mathbf{K_s}$-valued (probability) usual Gaussian measure.*

Proof. Let μ be a nontrivial usual Gaussian $\mathbf{K_s}$-valued measure on $Bco(X)$. Then by the definition its characteristic functional $\hat{\mu}$ must be satisfying Conditions $2.5.(3,5)$ $\mathbf{U_s}$-valued function and $\lim_{|y|\to\infty}\hat{\mu}(y)=0$ for each $y\in X^*\setminus\{0\}$, where X^* is the topological conjugate space to X of all continuous $\mathbf{K}$-linear functionals $f:X\to\mathbf{K}$. Moreover, there exist a $\mathbf{K}$-bilinear functional g and a compact non-degenerate $\mathbf{K}$-linear operator $T:X^*\to X^*$ with $ker(T)=\{0\}$ and a marked vector $x_0\in X$ such that $\hat{\mu}_{x_0}(y)=f(g(Ty,Ty))$ for each $y\in X^*$, where $\mu_{x_0}(dx):=\mu(-x_0+dx)$, $x\in X$. Since $\mathbf{K}$ is locally compact, then X^* is nontrivial and separates points of X (see [NB85, Roo78]). Each one-dimensional over $\mathbf{K}$ projection of a Gaussian measure is a Gaussian measure and products of Gaussian measures are Gaussian measures, hence convolutions of Gaussian measures are also Gaussian measures. Therefore, $\hat{\mu}_{x_0}:X^*\to\mathbf{U_s}$ is a nontrivial character: $\hat{\mu}_{x_0}(y_1+y_2)=\hat{\mu}_{x_0}(y_1)\hat{\mu}_{x_0}(y_2)$ for each y_1 and y_2 in X^*. If $char(\mathbf{K})=0$ and $\mathbf{K}$ is a non-Archimedean field, then there exists a prime number p such that $\mathbf{Q_p}$ is the subfield of $\mathbf{K}$. Then $\hat{\mu}(p^ny)=(\hat{\mu}(y))^{p^n}$ for each $n\in\mathbf{Z}$ and $y\in X^*\setminus\{0\}$, particularly, for $n\in\mathbf{N}$ tending to the infinity we have $\lim_{n\to\infty}p^ny=0$ and $\lim_{n\to\infty}\hat{\mu}_{x_0}(p^ny)=1$, $\lim_{n\to\infty}\hat{\mu}_{x_0}(y))^{p^n}=0$, since $s\neq p$ are primes, $\lim_{n\to\infty}\hat{\mu}_{x_0}(p^{-n}y)=0$ and $|\hat{\mu}_{x_0}(y)|<1$ for $y\neq 0$. This gives the contradiction, hence $\mathbf{K}$ can not be a non-Archimedean field of zero characteristic.

Suppose that $\mathbf{K}$ is a non-Archimedean field of characteristic $char(\mathbf{K})=p>0$, then $\mathbf{K}$ is isomorphic with the field of formal power series in variable t over a finite field $\mathbf{F_p}$. Therefore, $\hat{\mu}_{x_0}(py)=1$, but $\hat{\mu}_{x_0}(y)^p\neq 1$ for $y\neq 0$, since $\lim_{n\to\infty}\hat{\mu}_{x_0}(t^{-n}y)=0$. This contradicts the fact that $\hat{\mu}_{x_0}$ need to be the nontrivial character, consequently, $\mathbf{K}$ can not be a non-Archimedean field of nonzero characteristic as well. It remains the classical case of X over $\mathbf{R}$ or $\mathbf{C}$, but the latter case reduces to X over $\mathbf{R}$ with the help of the isomorphism of $\mathbf{C}$ as the $\mathbf{R}$-linear space with $\mathbf{R^2}$.

20. Theorem. *Let $\mu_{q,B,\gamma}$ and $\mu_{q,B,\delta}$ be two q-Gaussian $\mathbf{K_s}$-valued measures. Then $\mu_{q,B,\gamma}$ is equivalent to $\mu_{q,B,\delta}$ or $\mu_{q,B,\gamma}\perp\mu_{q,B,\delta}$ according to $v_q^s(\gamma-\delta)\in B^{1/2}(\mathsf{D}_{B,Y})$ or not. The measure $\mu_{q,B,\gamma}$ is orthogonal to $\mu_{g,B,\delta}$, when $q\neq g$. Two measures $\mu_{q,B,\gamma}$ and $\mu_{g,A,\delta}$ with positive definite nondegenerate A and B are either equivalent or orthogonal.*

21. Theorem. *The measures $\mu_{q,B,\gamma}$ and $\mu_{q,A,\gamma}$ are equivalent if and only if there exists a positive definite bounded invertible operator T such that $A=B^{1/2}TB^{1/2}$ and $T-I\in L_2(Y^*)$.*

Proof. Using the projective limit reduce consideration to the Banach space X. Let $z\in X$ be a marked vector and P_z be a projection operator on $z\mathbf{K}$ such that $P_z^2=P_z$, $z=\sum_j z_je_j$, then the characteristic functional of the projection $\mu_{q,B,\gamma}^{z\mathbf{K}}$ of $\mu_{q,B,\gamma}$ has the form $\hat{\mu}_{q,B,\gamma}^{z\mathbf{K}}=s^{[(\sum_{i,j}B_{i,j}v_q^s(z_i)v_q^s(z_j))v_{2q}^s(\xi)]}\chi_{\gamma(z)}(\xi)$ for each vector $x=\xi z$, where each z_j and $\xi\in\mathbf{K}$, since $v_{2q}^s(\xi)=(v_q^s(\xi))^2$. Choose a sequence $\{{}_nz:n\}$ in X such that it is the orthonormal basis in X and the operator $G:X\to X$ such that $G\,{}_nz={}_na\,{}_nz$ with ${}_na\neq 0$ for each $n\in\mathbf{N}$ and there exists $G^{-1}:G(X)\to X$ such that it induces the operator C on a dense subspace

$\mathcal{D}(Y)$ in Y such that $CBC : Y \to Y$ is invertible and $\|CBC\|$ and $\|(CBC)^{-1}\| \in [|\pi|, |\pi|^{-1}]$. Then $\mu_{q,A,\gamma}(dx)/\mu_{q,B,\gamma}(dx) = \lim_{n\to\infty}[\mu^{V_n}_{q,A,\gamma}(dx^n)/\lambda^{V_n}(dx^n)][\mu^{V_n}_{q,B,\gamma}(dx^n)/\lambda^{V_n}(dx^n)]^{-1}$, where $V_n := span_{\mathbf{K}}(\ _jz : j = 1,\dots,n)$, $x_n \in V_n$. Consider $x_n = G^{-1}(y_n)$, where $y_n \in G(V_n)$, then $[\mu^{V_n}_{q,B,\gamma}(G^{-1}dy^n)/\lambda^{V_n}(G^{-1}dy^n)]$ and $[\mu^{V_n}_{q,B,\gamma}(G^{-1}dy^n)/\lambda^{V_n}(G^{-1}dy^n)]^{-1}$ are in $L(\lambda^{V_n}(G^{-1}dy^n))$ for each n such that there exists $m \in \mathbf{N}$ for which $\|[\mu^{V_n}_{q,B,\gamma}(G^{-1}dy^n)/\lambda^{V_n}(G^{-1}dy^n)]\|$ and $\|[\mu^{V_n}_{q,B,\gamma}(G^{-1}dy^n)/\lambda^{V_n}(G^{-1}dy^n)]^{-1}\| \in [|\pi|, |\pi|^{-1}]$ for each $n > m$, where $\| * \|$ is taken in $L(\lambda^{V_n}(G^{-1}dy^n))$. Then $N_{\mu^{V_n}_{q,CBC,\gamma G^{-1}}}(y^n) \in [|\pi|, |\pi|^{-1}]$ for each $n > m$. Then the existence of $\mu_{q,A,\gamma}(dx)/\mu_{q,B,\gamma}(dx) \in L(\mu_{q,B,\gamma})$ is provided by using operator G and the consideration of characteristic functionals of measures, Theorem 3.5 and the fact that the Fourier-Stieltjes transform F is the isomorphism of Banach algebras $L(\mathbf{K}, Bco(\mathbf{K}), v, \mathbf{U_s})$ with $C_\infty(\mathbf{K}, \mathbf{U_s})$, where v denotes the Haar normalized by $v(B(\mathbf{K}, 0, 1)) = 1$ $\mathbf{K_s}$-valued measure on $\mathbf{K}$. If $g \neq q$ then the measure $\mu_{q,B,\gamma}$ is orthogonal to $\mu_{g,B,\delta}$, since

$$\lim_{R>0, R+n\to\infty} \sup_{x\in X^c_{R,n}} |(\mu_{q,B,\gamma})_{X_n}/(\mu_{g,B,\delta})_{X_n}|(x) = 0$$

for each $q > g$ due to Formula 4.(ii), where $X_n := span_{\mathbf{K}}(e_m : m = n, n+1, \dots, 2n)$, $X^c_{R,n} := X_n \setminus B(X_n, 0, R)$, $(\mu_{q,B,\gamma})_{X_n}$ is the projection of the measure $\mu_{q,B,\gamma}$ on X_n. Each term β_j in Theorem 3.5 is in $[0, 1] \subset \mathbf{R}$, consequently, the product in this theorem is either converging to a positive constant or diverges to zero, hence two measures $\mu_{q,B,\gamma}$ and $\mu_{g,A,\delta}$ are either equivalent or orthogonal.

2.7. Comments

1. In the article of W. Schikhof [Sch71] it was investigated the non-Archimedean analog of the Radon-Nikodym theorem. Let $(X, \mathcal{R}, \mu)$ be a measure space with a $\mathbf{K}$-valued measure μ and a covering ring $\mathcal{R}$ of X, where $\mathbf{K}$ is a non-Archimedean field complete relative to its nontrivial uniformity. If $1 \in L(X, \mathcal{R}, \mu, \mathbf{K})$, then N_μ is bounded on X. Let $\Omega := \{U \subset X : fCh_U \in L(X, \mathcal{R}, \mu, \mathbf{K})$ for each $f \in L(X, \mathcal{R}, \mu, \mathbf{K})\}$.

2. Definition. Let $\psi : \Omega \to \mathbf{K}$ be a function and μ be an integral corresponding to a measure μ and denoted by the same letter. Suppose $x \in X$, $a \in \mathbf{K}$, $r \in \mathbf{R}$.

(1). If for each $\varepsilon > 0$ there exists a neighborhood $V \in \mathcal{R}$ of x such that for all $U \subset V$, $U \in \Omega$ the inequality $|a - \psi(U)| < \varepsilon$ is satisfied, then we write $LIM_{U\to x}\psi(U) = a$.

(2). If for each $\varepsilon > 0$ there exists a neighborhood V of x such that for all $U \subset V$, $U \in \Omega$, with $|\mu(U)| \geq cN_\mu(x)$ we have $|a - \psi(U)| < \varepsilon$, then we write $LIM_{\mu,c;U\to x}\psi(U) = a$.

(3). $LIM_{\mu,U\to x}\psi(U) = a$ means $LIM_{\mu,c;U\to x}\psi(U) = a$ for each $c \in (0, 1)$.

(4). $\overline{LIM}_{U\to x}|\psi(U)| = r$ means, that for each $\varepsilon > 0$ there exists a neighborhood $V \in \mathcal{R}$ of x such that $r - \varepsilon \leq \sup\{|\psi(U)| : U \in \Omega, U \subset V\} \leq r + \varepsilon$.

3. Theorem. *Let μ be an integral on $L(X, \mathcal{R}, \mu, \mathbf{K})$ and let $f \in L(X, \mathcal{R}, \mu, \mathbf{K})$ and $x \in X$. Then*

(i). $LIM_{U\to x}(\mu(fCh_U) - f(x)\mu(U)) = 0$.

If $N_\mu(x) > 0$, then $LIM_{\mu,U\to x}\mu(fCh_U)\mu(U)^{-1} = f(x)$.

4. Theorem. *Let μ and ν be $\mathbf{K}$-valued measures on $(X, \mathcal{R})$. Then the following conditions are equivalent:*

(i). There exists $g \in L(\mu)$ *such that* $fg \in L(\mu)$ *for each* $f \in L(\nu)$ *and* $\nu(f) = \mu(fg)$.

(ii). For each $x \in X$ *for which* $N_\mu(x) > 0$ *there exists*

$$q(x) = LIM_{\mu,U\to x}\nu(U)\mu(U)^{-1}.$$

Moreover, every μ*-null set is* ν*-null.*

(iii). There exists a function $h : X \to \mathbf{K}$ *such that for each* $x \in X$*:* $LIM_{U\to a}[\nu(U) - h(x)\mu(U)] = 0$. *Further,* g, q, h *are uniquely determined up to* μ*-null functions and* $g = q = h$ μ*-almost everywhere. Finally,* $N_\nu = |g|N_\mu$.

5. Theorem. *Let* $\phi : X \to Y$ *be a surjective homeomorphism of* $(X, \mathcal{R}_X)$ *on* $(Y, \mathcal{R}_Y)$ *such that the mapping* $f \mapsto f \circ \phi$ *is a bijection of* $L(Y, \mathcal{R}_Y, \nu, \mathbf{K})$ *on* $L(X, \mathcal{R}_X, \mu, \mathbf{K})$*. Suppose that Conditions* $(1,2)$ *below are satisfied.*

(1)*. For* $x \in X$ *with* $N_\mu(x) > 0$ *there exists* $g(x) = LIM_{\mu,U\to x}\mu(U)^{-1}\nu(\phi(U))$.

(2)*. For each* $x \in X$*,* $N_\mu(x) = 0$ *implies* $N_\nu(\phi(x)) = 0$.

Then g is μ*-integrable and for every* $f \in L(\nu)$ *we have* $\nu(f) = \mu((f \circ \phi)g)$.

6. Remark. In § § 2.31, 3.9 and 3.12 the specific definitions of absolute continuity and quasi-invariance of $\mathbf{K_s}$-valued measures were given in view of Radon-Nikodym Theorem 4. The latter Theorem 5 serves for substitution of variables in the integral.

The fields $\mathbf{U_s}$ used above were investigated, for example, in [Dia84] (see also [Esc95, Roo78] and references therein).

7. Definitions. Let $(\mathbf{K}_j : j \in J)$ be a family of fields supplied with a non-Archimedean normalization. Consider a sub-ring $\overline{\prod_{j\in J}}\mathbf{K}_j$ of the product ring $\prod_{j\in J}\mathbf{K}_j$ formed by elements $a = (a_j : j \in J) \in \prod_{j\in J}\mathbf{K}_j$ such that $\sup_{j\in J}|a_j| < \infty$. Let $\mathcal{U}$ be an ultrafilter on the set J. We define an ultra-metric semi-normalization on $\overline{\prod_{j\in J}}\mathbf{K}_j$ such that $|a| := \lim_{\mathcal{U}}|a_j|$. The subset formed of all $a \in \overline{\prod_{j\in J}}\mathbf{K}_j$ with $|a| = 0$ forms the ideal denoted by $\mathcal{J}$.

We say that the quotient ring $[\overline{\prod_{j\in J}}\mathbf{K}_j]/\mathcal{J}$ supplied with the quotient absolute value is the ultra-product of fields $\mathbf{K}_j$ and it is denoted by $\overline{\prod_{j\in J}}\mathbf{K}_{j/\mathcal{U}}$ also. Denote also by a and $|a|$ elements in $\overline{\prod_{j\in J}}\mathbf{K}_{j/\mathcal{U}}$ and their absolute values.

Remind that an ultrafilter $\mathcal{U}$ on a set J is ω-incomplete, if there exists a sequence $(X_n : n \geq 0)$, $X_n \in \mathcal{U}$, $X_{n+1} \subset X_n$ for each $n \geq 0$ such that $\bigcap_{n\geq 0} X_n = \emptyset$.

8. Theorem. *Let* $(\mathbf{K}_j : j \in J)$ *be a family of fields with discrete valuations* v_j *such that* $v_j(\mathbf{K}_j) = \mathbf{Z}$ *and let* $(\rho_j : j \in J)$ *be a family of real numbers such that* $0 < \rho_j < 1$ *for each* j*. Consider in each* $\mathbf{K}_j$ *an absolute value* $|a_j| := \rho_j^{v_j(a_j)}$.

(1)*. If* $\lim_{\mathcal{U}}\rho_j = 0$*, then the absolute value in* $\overline{\prod_{j\in J}}\mathbf{K}_{j/\mathcal{U}}$ *is trivial.*

(2)*. If* $\lim_{\mathcal{U}}\rho_j = 1$*, then the field* $\mathbf{K} := \overline{\prod_{j\in J}}\mathbf{K}_{j/\mathcal{U}}$ *has the dense normalization such that* $\Gamma_{\mathbf{K}} = (0,\infty)$*. Moreover,* $\mathbf{K}$ *is* ω*-incomplete and spherically complete.*

(3)*. If* $0 < \lim_{\mathcal{U}}\rho_j = \rho < 1$*, then* $\overline{\prod_{j\in J}}\mathbf{K}_{j/\mathcal{U}}$ *is the field with the discrete valuation.*

(4)*. If each* $\mathbf{K}_j$ *is algebraically complete, then* $\mathbf{K}$ *is algebraically complete.*

9. Proposition. *Let* $\mathbf{L}$ *be a field with a discrete normalization group* $\Gamma_{\mathbf{L}}$*. Consider a family* $(\mathbf{K}_j : j \in J)$ *of all finite extensions of* $\mathbf{L}$ *such that* $[\mathbf{K}_j : \mathbf{L}] = [\mathbf{k}_j : \mathbf{l}] = n_j$*, where* $\mathbf{k}_j$ *is the residue class field of* $\mathbf{K}_j$ *and* $\mathbf{l}$ *is that of* $\mathbf{L}$.

(1)*. The field* $\mathbf{K} := \overline{\prod_{j\in J}}\mathbf{K}_{j/\mathcal{U}}$ *is a complete field with a discrete valuation and with the residue class field* $\mathbf{k} := \prod_{j\in J}\mathbf{k}_{j/\mathcal{U}}$.

(2)*. If* $\lim_{\mathcal{U}} n_j = \infty$ *and if* $\mathbf{l}$ *is perfect, then the field* $\mathbf{K} := \overline{\prod_{j\in J}}\mathbf{K}_{j/\mathcal{U}}$ *is the transcendental extension of* $\mathbf{L}_o := \overline{\prod_{j\in J}}\mathbf{L}_{/\mathcal{U}}$.

Chapter 3

Algebras of Real Measures on Groups

3.1. Introduction

Besides Banach spaces quasi-invariant measures were constructed on non-locally compact topological groups. For example, on a group of diffeomorphisms they were constructed for real locally compact manifolds M in [Kos94, Sha89] and for non-locally compact real or non-Archimedean manifolds M in [Lud96, Lud99t, Lud99r, Lud0348]. Such groups are also Banach manifolds or strict inductive limits of their sequences. Then on a real and non-Archimedean wrap (particularly loop) groups and semigroups of families of mappings from one manifold into another they were elaborated in [Lud98s, Lud00d, Lud02b, Lud00a, Lud08]. Then each Banach space over a locally compact field supplies an example of the additive group and quasi-invariant real-valued measures on it were described in Chapter 1. On real and non-Archimedean Banach-Lie groups quasi-invariant measures were constructed in [DS69, Lud0348].

This chapter is devoted to the investigation of properties of quasi-invariant measures that are important for analysis on topological groups and for construction irreducible representations [Kos94, Ner88]. Algebras of quasi-invariant measures and functions are defined and studied on topological groups. The following properties are investigated:

(1) convolutions of measures and functions,

(2) continuity of functions of measures,

(3) non-associative algebras generated with the help of quasi-invariant measures. The theorems given below show that many differences appear to be between locally compact and non-locally compact groups. The topological Hausdorff groups considered below are supposed to have structure of Banach manifolds over the corresponding fields if something another is not specified.

3.2. Algebras of Measures and Functions

1. Definitions. (a). Let G be a Hausdorff separable topological group. A real (or complex) Radon measure μ on $Af(G,\mu)$ is called left-quasi-invariant (or right) relative to a dense subgroup H of G, if $\mu_\phi(*)$ (or $\mu^\phi(*)$) is equivalent to $\mu(*)$ for each $\phi \in H$, where $Bf(G)$ is the Borel σ-field of G, $Af(G,\mu)$ is its completion by μ, $\mu_\phi(A) := \mu(\phi^{-1}A)$, $\mu^\phi(A) := \mu(A\phi^{-1})$ for each $A \in Af(G,\mu)$, $\rho_\mu(\phi,g) := \mu_\phi(dg)/\mu(dg)$ (or $\tilde{\rho}_\mu(\phi,g) := \mu^\phi(dg)/\mu(dg)$) denote a left (or right) quasi-invariance factor. We assume that a uniformity τ_G on G is such that $\tau_G|H \subset \tau_H$, (G,τ_G) and (H,τ_H) are complete. We suppose also that there exists an open base in $e \in H$ such that their closures in G are compact (such pairs exist for loop groups and groups of diffeomorphisms and Banach-Lie groups). We denote by $M_l(G,H)$ (or $M_r(G,H)$) a set of left-(or right) quasi-invariant measures on G relative to H with a finite norm $\|\mu\| := \sup_{A\in Af(G,\mu)} |\mu(A)| < \infty$.

(b). Let $L^p_H(G,\mu,\mathbf{C})$ for $1 \le p \le \infty$ denotes the Banach space of functions $f : G \to \mathbf{C}$ such that $f_h(g) \in L^p(G,\mu,\mathbf{C})$ for each $h \in H$ and

$$\|f\|_{L^p_H(G,\mu,\mathbf{C})} := \sup_{h\in H} \|f_h\|_{L^p(G,\mu,\mathbf{C})} < \infty,$$

where $f_h(g) := f(h^{-1}g)$ for each $g \in G$. For $\mu \in M_l(G,H)$ and $\nu \in M(H)$ let

$$(\nu * \mu)(A) := \int_H \mu_h(A)\nu(dh) \text{ and } (q\tilde{*}f)(g) := \int_H f(hg)q(h)\nu(dh)$$

be convolutions of measures and functions, where $M(H)$ is the space of Radon measures on H with a finite norm, $\nu \in M(H)$ and $q \in L^s(H,\nu,\mathbf{C})$, that is

$$\left(\int_H |q(h)|^s|\nu|(dh)\right)^{1/s} =: \|q\|_{L^s(H,\nu,\mathbf{C})} < \infty \text{ for } 1 \le s < \infty.$$

2. Lemma. *The convolutions are continuous* $\mathbf{C}$*-linear mappings*

$$* : M(H) \times M_l(G,H) \to M_l(G,H) \textit{ and}$$

$$\tilde{*} : L^1(H,\nu,\mathbf{C}) \times L^1_H(G,\mu,\mathbf{C}) \to L^1_H(G,\mu,\mathbf{C}).$$

Proof. It follows immediately from the definitions, Fubini theorem and because $d_\mu(h,g) \in L^1(H \times G, \nu \times \mu, \mathbf{C})$. In fact one has,

$$\|\nu * \mu\| \le \|\nu\| \times \|\mu\|, \ \|q\tilde{*}f\|_{L^1_H(G,\mu,\mathbf{C})} \le \|q\|_{L^1(H,\nu,\mathbf{C})} \times \|f\|_{L^1_H(G,\mu,\mathbf{C})}.$$

3. Definition. For $\mu \in M(G)$ its involution is given by the following formula: $\mu^*(A) := \overline{\mu(A^{-1})}$, where $\bar{b}$ denotes complex conjugated $b \in \mathbf{C}$, $A \in Af(G,\mu)$.

4. Lemma. *Let* $\mu \in M_l(G,H)$ *and* G *and* H *be non-locally compact with structures of Banach manifolds. Then* μ^* *is not equivalent to* μ.

Proof. Let $T : G \to TG$ be the tangent mapping. Then μ induces quasi-invariant measure λ from an open neighborhood W of the unit $e \in G$ on a neighborhood of the zero section V in T_eG and then it has an extension onto the entire T_eG. Let at first T_eG be a Hilbert space. Put $Inv(g) = g^{-1}$ then $T \circ Inv \circ T^{-1} =: K$ on V is such that there is not any operator B of trace class on T_eG such that $\tilde{M}_\lambda \subset B^{1/2}T_eG$ and $KT_eG \subset \tilde{M}_\lambda$, where $Re(1-\theta(z)) \to 0$ for $(Bz,z) \to 0$ and $z \in T_eG$, $\theta(z)$ is the characteristic functional of λ, $\tilde{M}_\lambda$ is a set of all $x \in T_eG$ such that λ_x is equivalent to λ (see Theorem 19.1 [Sko74]). Then using theorems for induced measures from a Hilbert space on a Banach space [DF91, Kuo75], we get the statement of Lemma 4.

5. Lemma. *For $\mu \in M_l(G,H)$ and $1 \le p < \infty$ the translation map $(q, f) \to f_q(g)$ is continuous from $H \times L^p_H(G,\mu,\mathbf{C})$ into $L^p_H(G,\mu,\mathbf{C})$.*

Proof. For metrizable G in view of the Lusin theorem (2.3.5 in [Fed69]) and definitions of τ_G and τ_H for each $\varepsilon > 0$ there are a neighborhood $V \ni e$ in H and compacts K_1 and K in G such that the closure $cl_G V K_1 =: K_2$ is compact in G with $K_2 \subset K$, a restriction $f|_{K_2}$ is continuous, $(|\tilde{\mu}| + |\mu|)(G \setminus K_2) < \varepsilon$, where $\tilde{\mu}(dg) := f(g)\mu(dg)$.

6. Proposition. *For a probability measure $\mu \in M(G)$ there exists an approximate unit, that is a sequence of non-negative continuous functions $\psi_i : G \to \mathbf{R}$ such that $\int_G \psi_i(g)\mu(dg) = 1$ and for each neighborhood $U \ni e$ in G there exists i_0 such that $supp(\psi_i) \subset U$ for each $i > i_0$.*

Proof. A finite union of compact subsets in G is compact. So in view of the Radon property of μ for each $b > 0$ and for each symmetric $U = U^{-1}$ neighborhood U of e in G there exist a compact set C_b in G such that $e \in C_b$ and $\mu(U \setminus C_b) < b\mu(U)$ and $\mu(G \setminus C_b) < b$. The group G has the structure of the Banach manifold also, hence its base of open neighborhoods of the unit element e is countable.

Choose such base $\{U_j : j\}$ with $U_j = U_j^{-1}$ and $cl(U_{j+1}) \subset U_j$ for each $j \in \mathbf{N}$, with $\bigcap_j U_j = \{e\}$. Then we get the sequence $b_j = 2^{-j}$ and $U = U_j$ and the corresponding compact subsets C_{b_j} in G. Take continuous non-negative functions ϕ_j with $supp(\phi_j) \subset U_j$ so that $\phi_j(g) = k_j$ for each $g \in C_{b_{j+1}} \cap U_{j+1}$, where $k_j \ge 1$ is a positive constant. This is possible due to the Uryson theorem (see § 1.5.10 in [Eng86]). This theorem states that if A and B are two disjoint closed subsets in a normal topological space S, then there exists a continuous function $f : S \to [0,1]$ such that $f(A) = \{1\}$ and $f(B) = \{0\}$.

Therefore, $0 < \mu(\phi_j) - k_j\mu(C_{b_{j+1}} \cap U_{j+1}) < k_j 2^{-j-1}$, so there exists a positive constant multiplier h_j so that $\psi_j := h_j\phi_j$ satisfies the equality $\mu(\psi_j) = 1$, certainly $supp(\psi_j) \subset U_k$ for each $j \ge k$ by the given construction.

7. Proposition. *If $(\psi_i : i \in \mathbf{N})$ is an approximate unit in H relative to a probability measure $\nu \in M(H)$, then $\lim_{i\to\infty} \psi_i \tilde{*} f = f$ in the $L^1_H(G,\mu,\mathbf{C})$ norm, where $\mu \in M_l(G,H)$, $f \in L^1_H(G,\mu,\mathbf{C})$.*

Proof. Given $b > 0$ we choose a neighborhood U of e in G such that $\|f(x) - f(e)\|_{L^1_H} < b$ for each $x \in U$. The mapping $x \mapsto f(xy)$ is continuous from $H \times L^1_H$ into L^1_H. If j is large enough so that $supp(\psi_j) \subset V$, where $V = V^{-1}$ is open and symmetric neighborhood of e in G and $V^2 \subset U$, then

$$\left\| \int_H f(yx)\psi_j(y)\nu(dy) - f(e) \right\|_{L^1_H} = \left\| \int_H [f(yx)\psi_j(y) - f(e)]\nu(dy) \right\|_{L^1_H}$$

$$\le \int_{U\cap H} \psi_j(y)\|f(yx) - f(e)\|_{L^1_H}\nu(dy) \le b\int_{U\cap H} \psi_j(y)\nu(dy) \le b,$$

where the convolution with ψ_j is the operator from L^1_H into L^1_H. In view of Lemmas 2, 5 we get the statement of this proposition.

8. Lemma. *Suppose $g \in L^q_H(G,\mu,\mathbf{C})$ and $(g^x|_H) \in L^q(H,\nu,\mathbf{C})$ for each $x \in G$, $f \in L^p(H,\nu,\mathbf{C})$ with $1 < p < \infty$, $1/p + 1/q = 1$, where $g^x(y) := g(yx)$ for each x and $y \in G$. Let μ and ν be probability measures, $\mu \in M_l(G,H)$, $\nu \in M(H)$. Then $f \tilde{*} g \in L^1_H(G,\mu,\mathbf{C})$ and* *there exists a function $h : G \to \mathbf{C}$ such that $h|_H$ is continuous, $h = f \tilde{*} g$ μ-a.e. on G and h vanishes at ∞ on G.*

Proof. In view of Fubini theorem and Hölder inequality we have

$$\| f \tilde{*} g\|_{L^1_H(G,\mu,\mathbf{C})} = \sup_{s\in H} \int_G \int_H |f(y)| \times |g(z)| \nu(dy)\mu((ys)^{-1}dz) \le$$

$$\sup_{s\in H} \left(\int_G \int_H |g(z)|^q \nu(dy)\mu((ys)^{-1}dz) \right)^{1/q} \times \left(\int_G \int_H |f(y)|^p \nu(dy)\mu((ys)^{-1}dz) \right)^{1/p} \le$$

$$\|f\|_{L^p(H,\nu,\mathbf{C})} \times \|g\|_{L^q_H(G,\mu,\mathbf{C})} \times \nu(H)\mu(G).$$

The equation $\alpha_f(\phi) := \int_H f(y)\overline{\phi(y)}\nu(dy)$ defines a continuous linear functional on $L^q(H,\nu,\mathbf{C})$. In view of Lemma 5 the function $\alpha_f(g^{(sx)^{-1}}) =: \tilde{h}((sx)^{-1}) =: w(s,x)$ of two variables s and x is continuous on $H \times H$ for $s,x \in H$, since the mapping $(s,x) \mapsto (sx)^{-1}$ is continuous from $H \times H$ into H. By Fubini theorem (see § 2.6.2 in [Fed69])

$$\int_G h(y)\psi(y)\mu(dy) = \int_G \int_H f(y)g(yx)\psi(x)\nu(dy)\mu(dx)$$

$$= \int_H f(y) \left[\int_G g(yx)\psi(x)\mu(dx) \right] \nu(dy)$$

for each $\psi \in L^p(G,\mu,\mathbf{C})$, since

$$\int_G \int_H |f(y)g(yx)\psi(x)| \, |\nu|(dy) \, |\mu|(dx) < \infty,$$

where $|\nu|$ denotes the variation of the real-valued measure ν, $h(y) := \tilde{h}(y^{-1})$. Here ψ is arbitrary in $L^p(G,\mu,\mathbf{C})$, from this it follows, that $\mu(\{y : h(y) \ne (f\tilde{*}g)(y), y \in G\}) = 0$, since h and $(f\tilde{*}g)$ are μ-measurable functions due to Fubini theorem and the continuity of the composition and the inversion in a topological group. In view of Lusin theorem (see § 2.3.5 in [Fed69]) for each $\varepsilon > 0$ there are compact subsets $C \subset H$ and $D \subset G$ and functions $f' \in L^p(H,\nu,\mathbf{C})$ and $g' \in L^q_H(G,\mu,\mathbf{C})$ with closed supports $supp(f') \subset C$, $supp(g') \subset D$ such that $cl_G CD$ is compact in G,

$$\|f' - f\|_{L^p(H,\nu,\mathbf{C})} < \varepsilon \text{ and } \|g' - g\|_{L^q_H(G,\mu,\mathbf{C})} < \varepsilon,$$

since by the supposition of § 1 the group H has the base B_H of its topology τ_H, such that the closures $cl_G V$ are compact in G for each $V \in \mathsf{B}_H$. From the inequality

$$|h'(x) - h(x)| \le (\|f\|_{L^p(H,\nu,\mathbf{C})} + \varepsilon)\varepsilon + \varepsilon\|g\|_{L^q_H(G,\mu,\mathbf{C})}$$

it follows that for each $\delta > 0$ there exists a compact subset $K \subset G$ with $|h(x)| < \delta$ for each $x \in G \setminus K$, where $h'(x^{-1}) := \alpha_{f'}(g'^x)$.

9. Proposition. *Let $A,B \in Af(G,\mu)$, μ and ν be probability measures, $\mu \in M_l(G,H)$, $\nu \in M(H)$. Then the function $\zeta(x) := \mu(A \cap xB)$ is continuous on H and $\nu(yB^{-1} \cap H) \in L^1(H,\nu,\mathbf{C})$. Moreover, if $\mu(A)\mu(B) > 0$, $\mu(\{y \in G : yB^{-1} \cap H \in Af(H,\nu)$ and $\nu(yB^{-1} \cap H) > 0\}) > 0$, then $\zeta(x) \ne 0$ on H.*

Proof. Let $g_x(y) := Ch_A(y)Ch_B(x^{-1}y)$, then $g_x(y) \in L^q_H(G,\mu,\mathbf{C})$ for $1 < q < \infty$, where $Ch_A(y)$ is the characteristic function of A. In view of Propositions 6 and 7 there exists $\lim_{i\to\infty} \psi_i \tilde{*} g_x = g_x$ in $L^1_H(G,\mu,\mathbf{C})$. In view of § 7 and Lemma 8 $\zeta(x)|_H$ is continuous. There is the following inequality:

$$1 \geq \int_H \mu(A\cap xB)\nu(dx) = \int_H \int_G ch_A(y)ch_B(x^{-1}y)\mu(dy)\nu(dx).$$

In view of Fubini theorem there exists

$$\int_H Ch_B(x^{-1}y)\nu(dy) = \nu((yB^{-1})\cap H) \in L^1(G,\mu,\mathbf{C}), \text{ hence}$$

$$\int_H \mu(A\cap xB)\nu(dx) = \int_G \nu(yB^{-1}\cap H)ch_A(y)\mu(dy).$$

10. Corollary. *Let $A,B \in Af(G,\mu)$, $\nu \in M(H)$ and $\mu \in M_l(G,H)$ be probability measures. Then denoting $Int_H V$ the interior of a subset V of H with respect to τ_H, one has*

(i) $Int_H(AB)\cap H \neq \emptyset$, when

$$\mu(\{y\in G : \nu(yB\cap H) > 0\}) > 0;$$

(ii) $Int_H(AA^{-1}) \ni e$, when

$$\mu(\{y\in G : \nu(yA^{-1}\cap H) > 0\}) > 0.$$

Proof. $AB\cap H \supset \{x \in H : \mu(A\cap xB^{-1}) > 0\}$.

10.1. Corollary. *Let $\mu \in M_l(G,H)$, $\rho_\mu(h,z) \in L^1(H,\nu,\mathbf{C})\times L^1(G,\mu,\mathbf{C})$, then $\rho_\mu(h,z)$ is continuous $\nu\times\mu$-a.e. on $H\times G$.*

Proof. Recall the Lusin's theorem. If ϕ is a Borel regular non-negative measure on a metric space X or a Radon measure on a locally compact Hausdorff space X, a function f is ϕ-measurable with values in a separable metric space Y, $A \subset X$ and A is ϕ-measurable with $\phi(A) < \infty$, $b > 0$, then A contains a closed or compact respectively subset C_b such that $\phi(A\setminus C_b) < b$ and the restriction $f|_{C_b}$ of f on C_b is continuous (see also Theorem 2.3.5 in [Fed69]).

In view of the co-cycle condition

$$\rho_\mu(\phi\psi,g) := \mu_{\phi\psi}(dg)/\mu(dg)$$

$$= (\mu_{\phi\psi}(dg)/\mu_\phi(dg))(\mu_\phi(dg)/\mu(dg))$$

$$= \rho_\mu(\psi,\phi^{-1}g)\rho_\mu(\phi,g)$$

on ρ_μ and Corollary 10 above and the Lusin's theorem for each $\varepsilon > 0$ the quasi-invariance factor $\rho_\mu(h,g)$ is continuous on $H\times A_\varepsilon$, where A_ε is a compact subset in G, but $H_\varepsilon := H\cap G_\varepsilon$ is a neighborhood of e in H, where $G_\varepsilon := A_\varepsilon \circ A_\varepsilon^{-1}$. Since H is dense in G, then $\mu(G\setminus \bigcup_{n=1}^\infty \bigcup_{j=1}^\infty h_j G_{1/n}) = 0$, where $\{h_j : j \in \mathbf{N}\}$ is a countable subset in G. Therefore, $\rho_\mu(h,z)$ is continuous $\nu\times\mu$-a.e. on $H\times G$.

10.2. Corollary. *Let G be a locally compact group, $\rho_\mu(h,z) \in L^1(G\times G,\mu\times\mu,\mathbf{C})$, then $\rho_\mu(h,z)$ is $\mu\times\mu$-a.e. continuous on $G\times G$.*

11. Corollary. *Let $G = H$. If $\mu \in M_l(G,H)$ is a probability measure, then G is a locally compact topological group.*

Proof. Let us take $\nu = \mu$ and $A = C \cup C^{-1}$, where C is a compact subset of G with $\mu(C) > 0$, whence $\mu(yA) > 0$ for each $y \in G$ and inevitably $Int_G(AA^{-1}) \ni e$.

12. Lemma. *Let $\mu \in M_l(G,H)$ be a probability measure and G be non-locally compact. Then $\mu(H) = 0$.*

Proof. In the particular case of a real Hilbert space X if a non-negative quasi-invariant measure μ on it is such that $\mu(L) = 0$ for each finite dimensional subspace, then $\mu(M_\mu) = 0$, where $M_\mu := \{a \in X : \int_X \tilde{\rho}_\mu(a,x)\mu(dx) = \mu(X)\}$ (see Theorem 19.2 [Sko74]). On the other hand for each real Banach space Y their exists a dense Hilbert subspace X a quasi-invariant measure μ on which indices a quasi-invariant measure ν on the initial Banach space Y (see [GV61] and § I.4 in [Kuo75]).

On the other hand, a quasi-invariant measure on G relative to left shifts $L_h g := hg$ with $h \in G'$ from the dense subgroup G', where $g \in G$, induces a quasi-invariant measure on its tangent space T_eG, since G has the structure of the smooth Banach manifold (see also [Kl82, Bou76]).

Therefore the statement of this lemma follows from and Theorem 3.25 of Chapter 1 and the proof of Lemma 4, since the embedding $T_eH \hookrightarrow T_eG$ is a compact operator in the non-Archimedean case and of trace class in the real case. See also the papers about construction of quasi-invariant measures on the considered here groups [DS69, Sha89, Kos94, Lud96, Lud99t, Lud98s, Lud99r, Lud00a, Lud00d, Lud02b].

Indeed, the measure μ on G is induced by the corresponding measure ν on a Banach space Z for which there exists a local diffeomorphism $A : W \to V$, where W is a neighborhood of e in G and V is a neighborhood of 0 in Z. The measure μ on G is quasi-invariant relative to H. Therefore, the measure ν on U is quasi-invariant relative to the action of elements $\psi \in W' \subset W \cap H$ due to the local diffeomorphism A, that is, ν_ϕ is equivalent to ν for each $\phi := A\psi A^{-1}$, where $AW'A^{-1}U \subset V$, W' is an open neighborhood of e in H and U is an open neighborhood of 0 in Z, $\nu_\phi(E) := \nu(\phi^{-1}E)$, ϕ is an operator on Z such that it may be non-linear. The quasi-invariance factor $\rho_\nu(\phi, v)$ has expressions through $|det(\phi')|$ and the quasi-invariance factor $q_\nu(z,x)$ relative to linear shifts $z \in Z'$ given by theorems from § 26 [Sko74] in the real case and Theorem 3.25 of Chapter 1 in the non-Archimedean case:

$$\nu_\phi(dx)/\nu(dx) = |det\{\phi'(\phi^{-1}(x))\}| q_\nu(x - \phi^{-1}(x), x),$$

where $x \in U$, $\phi = A\psi A^{-1}$, $\psi \in W'$. Then $(A\psi A^{-1}v - v) \in Z'$ for each $v \in V$ and $\psi \in W'$, where ν on Z is quasi-invariant relative to shifts on vectors $z \in Z'$ and there exists a compact operator in the non-Archimedean case and an operator of trace class in the real case of embedding $\theta : Z' \hookrightarrow Z$ such that $\nu(Z') = 0$.

13. Theorem. *Let (G, τ_G) and (H, τ_H) be a pair of topological non-locally compact groups G, H (Banach-Lie, Frechet-Lie or groups of diffeomorphisms or loop groups) with uniformities τ_G, τ_H such that H is dense in (G, τ_G) and there is a probability measure $\mu \in M_l(G,H)$ with continuous $d_\mu(z,g)$ on $H \times G$. Also let X be a Hilbert space over $\mathbf{C}$ and $U(X)$ be the unitary group. Then (1) if $T : G \to U(X)$ is a weakly continuous representation, then there exists $T' : G \to U(X)$ equal μ-a.e. to T and $T'|_{(H,\tau_H)}$ is strongly continuous;*

(2) if $T : G \to U(X)$ is a weakly measurable representation and X is separable, then there exists $T' : G \to U(X)$ equal to T μ-a.e. and $T'|_{(H,\tau_H)}$ is strongly continuous.

Proof. Let $R(G) := (I) \cup L^1(G,\mu)$, where I is the unit operator on L^1. Then we can define

$$A_{(\lambda e+a)_h} := \lambda I + \int_G a_h(g)[\rho_\mu(h^{-1},g)]T_g \mu(dg),$$

where $a_h(g) := a(h^{-1}g)$. Then

$$|(A_{(\lambda e+a)_h} - A_{\lambda e+a}\xi,\eta)| \leq \int_G |a_h(g)\rho_\mu(h,g) - a(g)|\,|(T_g\xi,\eta)|\mu(dg),$$

hence A_{a_h} is strongly continuous with respect to $h \in H$, that is,

$$\lim_{h\to e} |A_{a_h}\xi - A_a\xi| = 0.$$

Denote $A_{a_h} = T'_h A_a$ (see also § 29 [Nai68]), so $T'_h\xi = A_{a_h}\xi$, where $\xi = A_a\xi_0$, $a \in L^1$. Whence

$$(T'_h\xi, T'_h\xi) = (A_{a_h}\xi_0, A_{a_h}\xi_0) =$$

$$\int_G \bar{a}_h(g)(T_g\xi_0, T_{g'}\xi_0)\rho_\mu(h^{-1},g)\rho_\mu(h^{-1},g')a_h(g')\mu(dg)\mu(dg')$$

$$= \int_G \bar{a}(z)a(z')(U_z\xi_0, U_{z'}\xi_0)\mu(dz)\mu(dz') = (\xi,\xi).$$

Therefore, T'_h is uniquely extended to a unitary operator in the Hilbert space $X' \subset X$. In view of lemma 12, $\mu(H) = 0$. Hence T' may be considered equal to T μ-a.e. Then a space $span_{\mathbf{C}}[A_{a_h} : h \in H]$ is evidently dense in X, since

$$(A_{a_h}\xi_1, A_{x_q}\xi_0) = \left(\int_G a_h(g)T_g\rho_\mu(h^{-1},g)\mu(dg)\xi_1, \int_G x_q(g')T_{g'}\rho_\mu(q^{-1},g')\mu(dg')\xi_0 \right) =$$

$$= \left(T_h \int_G a(g)T_g\mu(dg)\xi_1, T_q \int_G x(g')T_{g'}\mu(dg') \right) = (T_{q^{-1}h}A_a\xi_1, A_x\xi_0).$$

For proving the second statement let

$$\mathsf{R} := [\xi : A_a\xi = 0 \text{ for each } a \in L^1(G,\mu)].$$

If

$$(A_a\xi,\eta) = \int_G a(g)(T_g\xi,\eta)\mu(dg) = \int_G a(g)(T'_g\xi,\eta)\mu(dg)$$

for each $a(g) \in L^1(G,\mu,\mathbf{C})$, then $(T_g\xi,\eta) = (T'_g\xi,\eta)$ for μ-almost all $g \in G$.

Suppose that $\{\xi_n : n \in \mathbf{N}\}$ is a complete orthonormal system in X. If $\xi \in X$, then

$$\int_G a(g)(T_g\xi,\xi_m)\mu(dg) = 0$$

for each $g \in G \setminus S_m$, where $\mu(S_m) = 0$. Therefore, $(T_g\xi,\xi_m) = 0$ for each $m \in \mathbf{N}$, if $g \in G \setminus S$, where $S := \bigcup_{m=1}^{\infty} S_m$. Hence $T_g\xi = 0$ for each $g \in G \setminus S$, consequently, $\xi = 0$. Then $(T_g\xi_n,\xi_m) = (T'_g\xi_n,\xi_m)$ for each $g \in G \setminus \gamma_{n,m}$, where $\mu(\gamma_{n,m}) = 0$. Hence $(T_g\xi_n,\xi_m) = (T'_g\xi_n,\xi_m)$ for each $n,m \in \mathbf{N}$ and each $g \in G \setminus \gamma$, where $\gamma := \bigcup_{n,m}\gamma_{n,m}$ and $\mu(\gamma) = 0$. Therefore, $\mathsf{R} = 0$.

14. Definition and note. Let $\{G_i : i \in \mathbf{N_o}\}$ be a sequence of topological groups such that $G = G_0$, $G_{i+1} \subset G_i$ and G_{i+1} is dense in G_i for each $i \in \mathbf{N_o}$ and their topologies are denoted τ_i, $\tau_i|_{G_{i+1}} \subset \tau_{i+1}$ for each i, where $N_o := \{0,1,2,\dots\}$. Suppose that these groups are supplied with real probability quasi-invariant measures μ^i on G_i relative to G_{i+1}. For example, such sequences exist for groups of diffeomorphisms or wrap (particularly loop) groups considered in previous papers [DS69, Sha89, Kos94, Lud96, Lud99t, Lud98s, Lud99r, Lud00a, Lud00d, Lud02b, Lud08].

Let $L^2_{G_{i+1}}(G_i,\mu^i,\mathbf{C})$ denotes a subspace of $L^2(G_i,\mu^i,\mathbf{C})$ as in § $1(b)$. Such spaces are Banach and not Hilbert in general. Let $\tilde{L}^2(G_{i+1},\mu^{i+1},L^2(G_i,\mu^i,\mathbf{C})) := H_i$ denotes the subspace of $L^2(G_i,\mu^i,\mathbf{C})$ of elements f such that

$$\|f\|_i^2 := [\|f\|^2_{L^2(G_i,\mu^i,\mathbf{C})} + \|f\|'^2_i]/2 < \infty, \text{ where}$$

$$\|f\|'^2_i := \int_{G_{i+1}} \int_{G_i} |f(y^{-1}x)|^2 \mu^i(dx)\mu^{i+1}(dy).$$

Evidently H_i are Hilbert spaces due to the parallelogram identity. Let

$$f^{i+1} * f^i(x) := \int_{G_{i+1}} f^{i+1}(y) f^i(y^{-1}x)\mu^{i+1}(dy)$$

denotes the convolution of $f^i \in H_i$.

15. Lemma. *The convolution* $* : H_{i+1} \times H_i \to H_i$ *is the continuous bilinear mapping.*

Proof. In view of Fubini theorem and Cauchy inequality:

$$\int_{G_{i+1}} \int_{G_i} |f^{i+1} * f^i(z^{-1}x)|^2 \mu^i(dx)\mu^{i+1}(dz)$$

$$= \left\{ \int_{G_{i+1}} \int_{G_i} \int_{G_{i+1}} f^{i+1}(y) f^i(y^{-1}z^{-1}x)\mu^{i+1}(dy) \right.$$

$$\left. \int_{G_{i+1}} \bar{f}^{i+1}(q) \bar{f}^i(q^{-1}z^{-1}x)\mu^{i+1}(dq)\mu^i(dx)\mu^{i+1}(dz) \right\}$$

$$\leq \int_{G_i} \int_{G_{i+1}} \left(\int_{G_{i+1}} |f^{i+1}(y)|^2 \mu^{i+1}(dy) \right)^{1/2} \left(\int_{G_{i+1}} |f^{i+1}(q)|^2 \mu^{i+1}(dq) \right)^{1/2}$$

$$\left(\int_{G_{i+1}} |f^i(y^{-1}z^{-1}x)|^2 \mu^{i+1}(dy) \right)^{1/2} \left(\int_{G_{i+1}} |f^i(q^{-1}z^{-1}x)|^2 \mu^{i+1}(dq) \right)^{1/2} \mu^i(dx)\mu^{i+1}(dz)$$

$$\leq \|f^{i+1}\|^2_{L^2(G_{i+1},\mu^{i+1},\mathbf{C})} \int_{G_i} \left[\int_{G_{i+1}} \int_{G_{i+1}} |f^i(y^{-1}z^{-1}x)|^2 \mu^{i+1}(dy)\mu^{i+1}(dz) \right]^{1/2}$$

$$\left[\int_{G_{i+1}} \int_{G_{i+1}} |f^i(q^{-1}z^{-1}x)|^2 \mu^{i+1}(dq)\mu^{i+1}(dz) \right]^{1/2} \mu^i(dx)$$

$$\leq \|f^{i+1}\|^2_{L^2(G_{i+1},\mu^{i+1},\mathbf{C})} \int_{G_{i+1}} \int_{G_i} \int_{G_{i+1}} |f^i(y^{-1}z^{-1}x)|^2 \mu^{i+1}(dy)\mu^{i+1}(dz)\mu^i(dx)$$

$$= \|f^{i+1}\|^2_{L^2(G_{i+1},\mu^{i+1},\mathbf{C})} \left(\int_{G_i} \int_{G_{i+1}} \int_{G_{i+1}} |f^i(y^{-1}\gamma)|^2 \mu^{i+1}(dy)\mu^{i+1}(dz) d_{\mu^i}(z^{-1},\gamma)\mu^i(d\gamma) \right)$$

$$\leq \| f^{i+1}\|^2_{L^2(G_{i+1},\mu^{i+1},\mathbf{C})} \int_{G_i}\int_{G_{i+1}} |f^i(z^{-1}x)|^2\mu^{i+1}(dz)\mu^i(dx),$$

since

$$\int_{G_i}\int_{G_{i+1}} d_{\mu^i}(z^{-1},\gamma)\mu^i(d\gamma)\mu^{i+1}(dz) = \int_{G_{i+1}}\mu^{i+1}(dz)\int_{G_i}\mu^i(zd\gamma) = 1.$$

Then

$$\| f^{i+1} * f^i\|^2_{L^2(G_i,\mu^i,\mathbf{C})} = \int_{G_i}\left|\int_{G_{i+1}} f^{i+1}(y)f^i(y^{-1}x)\mu^{i+1}(dy)\right|^2\mu^i(dx)$$

$$\leq \| f^{i+1}\|^2_{L^2(G_{i+1},\mu^{i+1},\mathbf{C})} \int_{G_i}\int_{G_{i+1}} |f^i(z^{-1}x)|^2\mu^{i+1}(dz)\mu^i(dx).$$

Therefore,

$$\| f^{i+1} * f^i\|_i \leq \| f^{i+1}\|_{L^2(G_{i+1},\mu^{i+1},\mathbf{C})}\| f^i\|_i.$$

16. Definition. Let $l_2(\{H_i : i \in \mathbf{N_o}\}) =: H$ be the Hilbert space consisting of elements $f = (f^i : f^i \in H_i, i \in \mathbf{N_o})$, for which

$$\| f\|^2 := \sum_{i=0}^{\infty} \| f^i\|_i^2 < \infty.$$

For elements f and $g \in H$ their convolution is defined by the formula: $f \star g := h$ with $h^i := f^{i+1} * g^i$ for each $i \in \mathbf{N_o}$. Let $* : H \to H$ be an involution such that $f^* := (\overline{f^j}^{\wedge} : j \in \mathbf{N_o})$, where $f^{j\wedge}(y_j) := f^j(y_j^{-1})$ for each $y_j \in G_j$, $f := (f^j : j \in \mathbf{N_o})$, $\bar{z}$ denotes the complex conjugated $z \in \mathbf{C}$.

17. Lemma. *H is a non-associative non-commutative Hilbert algebra with involution $*$, that is $*$ is conjugate-linear and $f^{**} = f$ for each $f \in H$.*

Proof. In view of Lemma 15 the convolution $h = f \star g$ in the Hilbert space H has the norm $\|h\| \leq \| f\|\ \|g\|$, hence is a continuous mapping from $H \times H$ into H. From its definition it follows that the convolution is bilinear. It is non-associative as follows from the computation of i-th terms of $(f \star g) \star q$ and $f \star (g \star q)$, which are $(f^{i+2} * g^{i+1}) * q^i$ and $f^{i+1} * (g^{i+1} * q^i)$ respectively, where f, g and $q \in H$. It is non-commutative, since there are f and $g \in H$ for which $f^{i+1} * g^i$ are not equal to $g^{i+1} * f^i$. Since $f^{j\wedge\wedge}(y_j) = f^j(y_j)$ and $\bar{\bar{z}} = z$, one has $f^{**} = (f^*)^* = f$.

18. Note. In general $(f \star g^*)^* \neq g \star f^*$ for f and $g \in H$, since there exist f^j and g^j such that $g^{j+1} * (f^j)^* \neq (f^{j+1} * (g^j)^*)^*$. If $f \in H$ is such that $f^j|_{G_{j+1}} = f^{j+1}$, then

$$((f^{j+1})^* * f^j)(e) = \int_{G_{j+1}} \bar{f}^{j+1}(y^{-1}) f^{j+1}(y)\mu^{j+1}(dy) = \| f^{j+1}\|^2_{L^2(G_{j+1},\mu^{j+1},\mathbf{C})},$$

where $j \in \mathbf{N_o}$.

19. Definition. Let $l_2(\mathbf{C})$ the standard Hilbert space over the field $\mathbf{C}$ be considered as a Hilbert algebra with the convolution $\alpha \star \beta = \gamma$ such that $\gamma^i := \alpha^{i+1}\beta^i$, where $\alpha := (\alpha^i : \alpha^i \in \mathbf{C}, i \in \mathbf{N_o})$, α, β and $\gamma \in l_2(\mathbf{C})$.

20. Note. The algebra $l_2(\mathbf{C})$ has two-sided ideals $J_i := \{\alpha \in l_2(\mathbf{C}) : \alpha^j = 0$ for each $j > i\}$, where $i \in \mathbf{N_o}$. That is, $J \star l_2(\mathbf{C}) \subset J$ and $l_2(\mathbf{C}) \star J = J$ and J is the $\mathbf{C}$-linear subspace of $l_2(\mathbf{C})$, but $J \star l_2(\mathbf{C}) \neq J$. There are also right ideals, which are not left ideals: $K_i := \{\alpha \in l_2(\mathbf{C}) : \alpha^j = 0$ for each $j = 0,\dots,i\}$, where $j \in \mathbf{N_o}$. That is, $l_2(\mathbf{C}) \star K_i = K_i$, but

$K_i \star l_2(\mathbf{C}) = K_{i-1}$ for each $i \in \mathbf{N_o}$, where $K_{-1} := l_2(\mathbf{C})$. The algebra $l_2(\mathbf{C})$ is the particular case of H, when $G_j = \{e\}$ for each $j \in \mathbf{N_o}$. We consider further H for non-trivial topological groups outlined above.

21. Theorem. *If F is a maximal proper left or right ideal in H, then H/F is isomorphic as the non-associative noncommutative algebra over* $\mathbf{C}$ *with* $l_2(\mathbf{C})$.

Proof. Since F is the ideal, it is the $\mathbf{C}$-linear subspace of H. Suppose, that there exists $j \in \mathbf{N_o}$ such that $f^j = 0$ for each $f \in F$, then $f^i = 0$ for each $i \in \mathbf{N_o}$, since the space of bounded complex-valued continuous functions $C^0_b(G_\infty, \mathbf{C})$ on $G_\infty := \bigcap_{j=0}^\infty G_j$ is dense in each $H_j := \{f^j : f \in H\}$ and $C^0_b(G_\infty, \mathbf{C}) \cap F_j = \{0\}$ and $C^0_b(G_j, \mathbf{C})|_{G_{j+1}} \supset C^0_b(G_{j+1}, \mathbf{C})$. Therefore, $F_j \ne \{0\}$ for each $j \in \mathbf{N_o}$, consequently, $\mathbf{C} \hookrightarrow F_j$ for each $j \in \mathbf{N_o}$. Since $\mathbf{C}$ is embeddable into each F_j, then there exists the embedding of $l_2(\mathbf{C})$ into F, where $H_j := \{f^j : f \in H\}$, $\pi_j : H \to H_j$ are the natural projections.

The subalgebra F is closed in H, since H is the topological algebra and F is the maximal proper subalgebra. The space $H_\infty := \bigcap_{j \in \mathbf{N_o}} H_j$ is dense in each H_j and the group $G_\infty := \bigcap_{j \in \mathbf{N_o}} G_j$ is dense in each G_j.

Suppose that $F_i = H_i$ for some $i \in \mathbf{N_o}$, then $F_j = H_j$ for each $j \in \mathbf{N_o}$, since $C^0_b(G_\infty, \mathbf{C})$ is dense in each H_j and $C^0_b(G_j, \mathbf{C})|_{G_{j+1}} \supset C^0_b(G_{j+1}, \mathbf{C})$. The ideal F is proper, consequently, $F_j \ne H_j$ as the $\mathbf{C}$-linear subspace for each $j \in \mathbf{N_o}$, where $F_j = \pi_j(F)$.

There are linear continuous operators from $l_2(\mathbf{C})$ into $l_2(\mathbf{C})$ given by the following formulas: $x \mapsto (0, \dots, 0, x^0, x^1, x^2, \dots)$ with 0 as n coordinates at the beginning, $x \mapsto (x^n, x^{n+1}, x^{n+2}, \dots)$ for $n \in \mathbf{N}$; $x \mapsto (x^{kl+\sigma_k(i)} : k \in \mathbf{N_o}, i \in (0, 1, \dots, l-1))$, where $\mathbf{N} \ni l \ge 2$, $\sigma_k \in S_l$ are elements of the symmetric group S_l of the set $(0, 1, \dots, l-1)$. Then $f \star (g \star h) + l_2(\mathbf{C})$ and $(f \star g) \star h + l_2(\mathbf{C})$ are considered as the same class, also $f \star g + l_2(\mathbf{C}) = g \star f + l_2(\mathbf{C})$ in $H/l_2(\mathbf{C})$, since $(f + l_2(\mathbf{C})) \star (g + l_2(\mathbf{C})) = f \star g + l_2(\mathbf{C})$ for each f, g and $h \in H$. For each $f, g, h \in F$: $f \star (g \star h) + l_2(\mathbf{C})$ and $(f \star g) \star h + l_2(\mathbf{C})$ are considered as the same class, also $f \star g + l_2(\mathbf{C}) = g \star f + l_2(\mathbf{C})$ in $F/l_2(\mathbf{C})$, since $(f + l_2(\mathbf{C})) \star (g + l_2(\mathbf{C})) = f \star g + l_2(\mathbf{C}) \subset F$ for each f and $g \in F$. Therefore, the quotient algebras $H/l_2(\mathbf{C})$ and $F/l_2(\mathbf{C})$ are the associative commutative Banach algebras.

Let us adjoint a unit to $H/l_2(\mathbf{C})$ and $F/l_2(\mathbf{C})$. As a consequence of the Gelfand and Mazur theorem we have, that $(H/l_2(\mathbf{C}))/(F/l_2(\mathbf{C}))$ is isomorphic with $\mathbf{C}$ (see also Theorem V.6.12 [FD88] and Theorem III.11.1 [Nai68]). On the other hand, as it was proved above $F_j \ne H_j$ for each $j \in \mathbf{N_o}$, hence there exists the following embedding $l_2(\mathbf{C}) \hookrightarrow (H/F)$ and $(H/F)/l_2(\mathbf{C})$ is isomorphic with $(H/l_2(\mathbf{C}))/(F/l_2(\mathbf{C}))$. Therefore, H/F is isomorphic with $l_2(\mathbf{C})$.

3.3. Comments

Another methods of construction of unitary representations of topological totally disconnected groups which may be non-locally compact with the help of quasi-invariant real-valued measures were given in [Lud98b, Lud02b, Lud00a, Lud99t, Lud01s, Lud0348, Lud01f, Lud08, LD03] and references therein.

Chapter 4

Algebras of Non-Archimedean Measures on Groups

4.1. Introduction

In Chapter II quasi-invariant measures on Banach spaces with values in fields supplied with non-Archimedean multiplicative norms were studied. Besides Banach spaces they were constructed and investigated on non-locally compact topological groups. Quasi-invariant measures with values in non-Archimedean fields on a group of diffeomorphisms were constructed for non-Archimedean manifolds M in [Lud96, Lud99t, Lud08]. On non-Archimedean wrap (particularly loop) groups and semigroups they were provided in [Lud98s, Lud00a, Lud02b, Lud08]. A Banach space over a locally compact field also serves as the additive group and quasi-invariant measures on it were studied in Chapter 2.

This chapter is devoted to the investigation of properties of quasi-invariant measures with values in non-Archimedean fields that are important for analysis on topological groups and for construction of irreducible representations. The following properties are investigated:

(1) convolutions of measures and functions,

(2) continuity of functions of measures,

(3) non-associative algebras generated with the help of quasi-invariant measures. The theorems given below show that many differences appear to be between locally compact and non-locally compact groups. Algebras of measures and functions on groups are considered below. The groups considered below are supposed to have structure of Banach manifolds over the corresponding fields if something other is not specified.

4.2. Algebras of Measures and Functions

1. Definitions. (a). Let G be a Hausdorff separable topological group. A tight measure μ on $Af(G,\mu)$ with values in a non-Archimedean field $\mathbf{F}$ is called left-quasi-invariant (or right) relative to a dense subgroup H of G, if $\mu_\phi(*)$ (or $\mu^\phi(*)$) is equivalent to $\mu(*)$ for each $\phi \in H$, where $Bco(G)$ is the algebra of all clopen subsets of G, $Af(G,\mu)$ denotes its completion by μ, $\mu_\phi(A) := \mu(\phi^{-1}A)$, $\mu^\phi(A) := \mu(A\phi^{-1})$ for each $A \in Af(G,\mu)$, $\rho_\mu(\phi,g) :=$

$\mu_\phi(dg)/\mu(dg) \in L(G, Af(G,\mu), \mu, \mathbf{F})$ (or $\tilde{\rho}_\mu(\phi, g) := \mu^\phi(dg)/\mu(dg)$) denotes a left (or right) quasi-invariance factor, $\mathbf{F}$ is a non-Archimedean field complete relative to its uniformity and such that $\mathbf{K}_s \subset \mathbf{F}$. We assume that a uniformity τ_G on G is such that $\tau_G|H \subset \tau_H$, (G,τ_G) and (H,τ_H) are complete. We suppose also that there exists an open base in $e \in H$ such that their closures in G are compact. Such pairs exist for wrap (particularly loop) groups and groups of diffeomorphisms and Banach-Lie groups. We denote by $M_l(G,H)$ (or $M_r(G,H)$) a set of left-(or right) quasi-invariant tight measures on G relative to H with a finite norm $\|\mu\| < \infty$.

(b). Let $L_H(G,\mu,\mathbf{F})$ denotes the Banach space of functions $f : G \to \mathbf{F}$ such that $f_h(g) \in L(G,\mu,\mathbf{F})$ for each $h \in H$ and

$$\|f\|_{L_H(G,\mu,\mathbf{F})} := \sup_{h\in H} \|f_h\|_{L(G,\mu,\mathbf{F})} < \infty,$$

where $\mathbf{F}$ is a non-Archimedean field for which $\mathbf{K}_s \subset \mathbf{F}$, $f_h(g) := f(h^{-1}g)$ for each $g \in G$. For $\mu \in M_l(G,H)$ and $\nu \in M(H)$ let

$$(\nu * \mu)(A) := \int_H \mu_h(A)\nu(dh) \text{ and } (q\tilde{*}f)(g) := \int_H f(hg)q(h)\nu(dh)$$

be convolutions of measures and functions, where $M(H)$ is the space of tight measures on H with a finite norm, $\nu \in M(H)$ and $q \in L(H,\nu,\mathbf{F})$.

2. Lemma. *The convolutions are continuous* $\mathbf{F}$*-linear mappings*

$$* : M(H) \times M_l(G,H) \to M_l(G,H) \text{ and}$$

$$\tilde{*} : L(H,\nu,\mathbf{F}) \times L_H(G,\mu,\mathbf{F}) \to L_H(G,\mu,\mathbf{F}).$$

Proof. If X and Y are sets with separating covering rings $\mathcal{R}$ and $\mathcal{S}$ and measures μ and ν on them, then the Banach space $L(\mu\times\nu)$ is linearly topologically isomorphic with the (Banach completed) tensor product $L(\mu)\hat{\otimes}L(\nu)$ of Banach spaces $L(\mu)$ and $L(\nu)$ (see Theorem 7.16 and Chapter 4 in [Roo78]). By the definition this means that if E and F are two Banach spaces over the same field and $E\hat{\otimes}F$ is their (Banach completed) tensor product then the mapping $E \times F \ni (x,y) \mapsto (x\otimes y) \in E\hat{\otimes}F$ is characterized by two conditions:

(Bi) $\|x\otimes y\| \le \|x\|\|y\|$ for all $x \in X$ and $y \in Y$;

(Bii) for each continuous bilinear mapping S of $E\times F$ into any Banach space H over the same field there exists a unique continuous linear bounded operator $S_\otimes$ with the norm $\|S_\otimes\| \le \|S\|$ and such that $S_\otimes \circ \theta(x,y) = S(x,y) \in H$ for all $x \in E$ and $y \in F$, where $\theta : E\times F \hookrightarrow E\hat{\otimes}F$ is the natural embedding.

In view of the aforementioned theorem and estimates

$$\|\nu*\mu\| \le \|\nu\|\times\|\mu\|,\ \|q\tilde{*}f\|_{L_H(G,\mu,\mathbf{F})} \le \|q\|_{L(H,\nu,\mathbf{F})} \times \|f\|_{L_H(G,\mu,\mathbf{F})},$$

since $\rho_\mu(h,g) \in L(H\times G, \nu\times\mu, \mathbf{F})$, we get the statement of this lemma.

3. Lemma. *For* $\mu \in M_l(G,H)$ *the translation map* $(q, f) \to f_q(g)$ *is continuous from* $H\times L_H(G,\mu,\mathbf{F})$ *into* $L_H(G,\mu,\mathbf{F})$.

Proof. In view of Lemma 7.10 and Theorem 7.12 [Roo78] recalled in § 2.32 above for each $\varepsilon > 0$ the set $\{x : |f(x)|N_\mu(x) \ge \varepsilon\}$ is $Af(G,\mu)$-compact and f is $Af(G,\mu)$-continuous.

The embedding of H into G is compact (see § 1), hence for each $q \in H$ there exists V clopen in H and such that $q^{-1}V$ is a subgroup of H with $cl_G q^{-1}V$ compact in G, where $cl_G(A)$ denotes the closure of a subset A in G.

The product of compact subsets in G is compact in G, hence $f_q(g) \in L_H(G,\mu,\mathbf{F})$ and $(q,f) \mapsto f_q(g)$ is the continuous mapping, since the restriction of the $Bco(G)$ and the $Af(G,\mu)$-topologies onto X_ε coincide, $\|f_q\|_{L_H(\mu)} = \|f\|_{L_H(\mu)}$ for each $q \in H$ (see § 1.(b)).

4. Proposition. *For a probability measure $\mu \in M(G)$ there exists an approximate unit which is a sequence of nonzero continuous functions $\psi_i : G \to \mathbf{F}$ such that $\int_G \psi_i(g)\mu(dg) = 1$ and for each neighborhood $U \ni e$ in G there exists i_0 such that $supp(\psi_i) \subset U$ for each $i > i_0$.*

Proof. A group G has a countable base of neighborhoods of $e \in G$. A measure μ is quasi-invariant, hence $\|U\|_\mu > 0$ for each neighborhood $U \ni e$, μ is the tight measure, hence there exists a system of neighborhoods $\{U_i : U_i \ni e \forall i\}$, $\bigcap_i U_i = \{e\}$, $U_i \supset U_{i+1}$ for each i, $supp(\psi_i) \subset U_i$. Choose ψ_i such that $\int_G \psi_i(g)\mu(dg) = 1$ for each i.

5. Proposition. *If $(\psi_i : i \in \mathbf{N})$ is an approximate unit in H relative to a probability measure $\nu \in M(H)$, then $\lim_{i\to\infty} \psi_i * f = f$ in the $L_H(G,\mu,\mathbf{F})$ norm, where $\mu \in M_l(G,H)$, $f \in L_H(G,\mu,\mathbf{F})$.*

Proof. In view of Theorem 7.12 [Roo78] recalled in § 2.32 above for each $\varepsilon > 0$ and each i there exists a finite number of $h_j \in H$, $j = 1,\dots,n$, $n \in \mathbf{N}$, such that $\bigcup_{j=1}^n h_j U_i \supset X_{\varepsilon,f} \times V$, where $\{x : |f(x)|N_\mu(x) \geq \varepsilon\} =: X_{\varepsilon,f}$, V is a clopen neighborhood of $\xi \in H$ which can be chosen such that $\xi^{-1}V$ is a subgroup of H, $cl_G(\xi^{-1}V)$ is compact in G, $(\psi_i * f)(g) = \int_{h\in U_i} \psi_i(h) f(hg)\nu(dh)$ and $\sup_{g\in G} |f_\xi(g) - (\psi_i * f_\xi)(g)|N_\mu(g) \leq \sup_{g\in G}[\sup_{h\in X_{\varepsilon,f_\xi}} |\psi_i(h)||f_\xi(hg) - f_\xi(g)|N_\mu(g) + \varepsilon\|\psi_i\|_\nu\|G\|_\mu]$, hence $\lim_{i\to\infty}(\psi_i * f) = f$ in $L_H(G,\mu,\mathbf{F})$-norm.

6. Lemma. *Suppose $g \in L_H(G,\mu,\mathbf{F})$ and $(g^x|_H) \in L(H,\nu,\mathbf{F})$ for each $x \in G$, $f \in L(H,\nu,\mathbf{F})$, where $g^x(y) := g(yx)$ for each x and $y \in G$. Let μ and ν be probability measures, $\mu \in M_l(G,H)$, $\nu \in M(H)$. Then $f \tilde{*} g \in L_H(G,\mu,\mathbf{F})$ and there exists a function $h : G \to \mathbf{F}$ such that $h|_H$ is continuous, $h = f \tilde{*} g$ μ-a.e. on G and h vanishes at ∞ on G.*

Proof. In view of the Fubini theorem we have

$$\|f\tilde{*}g\|_{L_H(G,\mu,\mathbf{F})} = \sup_{h\in H, z\in G} \left| \int_H f(y)g(yz)\nu(dy) \right| N_\mu(z)$$

$$\leq \|g(z)\|_{L_H(G,\mu,\mathbf{F})} \|f\|_{L(H,\nu,\mathbf{F})}$$

The equation

$$\alpha_f(\phi) := \int_H f(y)\phi(y)\nu(dy)$$

defines a continuous linear functional on the Banach space

$$L^\infty(H,\nu,\mathbf{F}) := \{\phi : H \to \mathbf{F} : \ \phi \text{ is } (Af(H,\nu), Bco(\mathbf{F}))\text{-measurable},$$

$$\|\phi\|_\infty := ess_\nu - \sup_{x\in H} |\phi(x)| < \infty\}.$$

In view of Lemma 3 the function

$$\alpha_f(g^{(qx)^{-1}}) =: v(qx) =: w(q,x)$$

of two variables q and x is continuous on $H \times H$ for $q,x \in H$, since the mapping $(q,x) \mapsto (qx)^{-1}$ is continuous from $H \times H$ into H. By Theorem 7.16 [Roo78] recalled in § 2

$$\int_G v(y)\psi(y)\mu(dy) = \int_G \int_H f(y)g(yx)\psi(x)\nu(dy)\mu(dx)$$

$$= \int_H f(y)\left[\int_G g(yx)\psi(x)\mu(dx)\right]\nu(dy)$$

for each $\psi \in L^\infty(G,\mu,\mathbf{F})$. From this it follows, that $\mu(\{y : h(y) \neq (f\tilde{*}g)(y), y \in G\}) = 0$, since h and $(f\tilde{*}g)$ are μ-measurable functions due to Fubini theorem and the continuity of the composition and the inversion in a topological group. In view of Theorem 7.12 [Roo78] recalled in § 2.32 for each $\varepsilon > 0$ there are compact subsets $C \subset H$ and $D \subset G$ and functions $f' \in L(H,\nu,\mathbf{F})$ and $g' \in L_H(G,\mu,\mathbf{F})$ with closed supports $supp(f') \subset C$, $supp(g') \subset D$ such that $cl_G CD$ is compact in G,

$$\|f' - f\|_{L(H,\nu,\mathbf{F})} < \varepsilon \text{ and } \|g' - g\|_{L_H(G,\mu,\mathbf{F})} < \varepsilon,$$

since by the supposition of § 1 the group H has the base B_H of its topology τ_H, such that the closures $cl_G V$ are compact in G for each $V \in \mathsf{B}_H$. From the inequality

$$|h'(x) - h(x)| \leq (\|f\|_{L(H,\nu,\mathbf{F})} + \varepsilon)\varepsilon + \varepsilon\|g\|_{L_H(G,\mu,\mathbf{F})}$$

it follows that for each $\delta > 0$ there exists a compact subset $K \subset G$ with $|h(x)| < \delta$ for each $x \in G \setminus K$, where $h'(x^{-1}) := \alpha_{f'}(g'^x)$.

7. Proposition. *Let $A,B \in Af(G,\mu)$, μ and ν be probability measures, $\mu \in M_l(G,H)$, $\nu \in M(H)$. Then the function $\zeta(x) := \mu(A \cap xB)$ is continuous on H and $\nu(yB^{-1} \cap H) \in L(H,\nu,\mathbf{F})$. Moreover, if $\|A\|_\mu \|B\|_\mu > 0$, $\mu(\{y \in G : yB^{-1} \cap H \in Af(H,\nu)$ and $\|yB^{-1} \cap H\|_\nu > 0\}) > 0$, then $\zeta(x) \neq 0$ on H.*

Proof. Let $g_x(y) := Ch_A(y)Ch_B(x^{-1}y)$, then $g_x(y) \in L_H(G,\mu,\mathbf{F})$, where $Ch_A(y)$ is the characteristic function of A. In view of Propositions 4 and 5 there exists $\lim_{i\to\infty} \psi_i * g_x = g_x$ in $L_H(G,\mu,\mathbf{F})$. In view of Lemma 6 $\zeta(x)|_H$ is continuous. There is the following inequality:

$$1 \geq \left|\int_H \mu(A \cap xB)\nu(dx)\right| = \left|\int_H \int_G Ch_A(y)Ch_B(x^{-1}y)\mu(dy)\nu(dx)\right|.$$

In view of Theorem 7.16 [Roo78] recalled in § 2 there exists

$$\int_H Ch_B(x^{-1}y)\nu(dy) = \nu((yB^{-1}) \cap H) \in L(G,\mu,\mathbf{F}), \text{ hence}$$

$$\int_H \mu(A \cap xB)\nu(dx) = \int_G \nu(yB^{-1} \cap H)Ch_A(y)\mu(dy).$$

8. Corollary. *Let $A,B \in Af(G,\mu)$, $\nu \in M(H)$ and $\mu \in M_l(G,H)$ be probability measures. Then denoting $Int_H V$ the interior of a subset V of H with respect to τ_H, one has*

(i) $Int_H(AB) \cap H \neq \emptyset$, when

$$\|\{y \in G : \|yB \cap H\|_\nu > 0\}\|_\mu > 0;$$

(*ii*) *$Int_H(AA^{-1}) \ni e$, when*

$$\|\{y \in G : \|yA^{-1} \cap H\|_\nu > 0\}\|_\mu > 0.$$

Proof. We infer the inclusion $AB \cap H \supset \{x \in H : \ \|(A \cap xB^{-1})\|_\mu > 0\}$ demonstrating this corollary.

9. Corollary. *Let $G = H$. If $\mu \in M_l(G,H)$ is a probability measure, then G is a locally compact topological group.*

Proof. Let us take $\nu = \mu$ and $A = C \cup C^{-1}$, where C is a compact subset of G with $\|(C)\|_\mu > 0$, whence $\|(yA)\|_\mu > 0$ for each $y \in G$ and inevitably $Int_G(AA^{-1}) \ni e$.

10. Corollary. *Let $\mu \in M_l(G,H)$, $\rho_\mu(h,z) \in L(H,\nu,\mathbf{F}) \times L(G,\mu,\mathbf{F})$, then $\rho_\mu(h,z)$ is continuous $\nu \times \mu$-a.e. on $H \times G$.*

Proof. In view of the co-cycle condition

$$\rho_\mu(\phi\psi,g) := \mu_{\phi\psi}(dg)/\mu(dg) = (\mu_{\phi\psi}(dg)/\mu_\phi(dg))(\mu_\phi(dg)/\mu(dg))$$
$$= \rho_\mu(\psi,\phi^{-1}g)\rho_\mu(\phi,g)$$

on ρ_μ and Corollary 8 above, Theorem 7.12 [Roo78] recalled in § 2.32 for each $\varepsilon > 0$ the quasi-invariance factor $\rho_\mu(h,g)$ is continuous on $H \times G_\varepsilon$, but G_ε is neighborhood of e in G, where $G_\varepsilon := \{g \in G : \ N_\mu(g) \geq \varepsilon\}$. Since H is dense in G and G is separable, then $\bigcup_{j=1}^{\infty} h_j G_\varepsilon = G$, where $\{h_j : j \in \mathbf{N}\}$ is a countable subset in G. Therefore, $\rho_\mu(h,z)$ is continuous $\nu \times \mu$-a.e. on $H \times G$.

11. Corollary. *Let G be a locally compact group, $\rho_\mu(h,z) \in L(G \times G, \mu \times \mu, \mathbf{F})$, then $\rho_\mu(h,z)$ is $\mu \times \mu$-a.e. continuous on $G \times G$.*

Proof. It follows from Corollaries 9 and 10.

12. Remark. The latter two corollaries show, that the condition of continuity of $\rho_\mu(h,z)$ imposed in Chapter 2 is not restrictive.

13. Lemma. *Let $\mu \in M_l(G,H)$ be a probability measure and G be non-locally compact. Then $\|H\|_\mu = 0$.*

Proof. This follows from Theorem 2.3.13 above and the proof of Lemma 2, since the embedding $T_eH \hookrightarrow T_eG$ is a compact operator in the non-Archimedean case and a tight measure μ on G induces a tight measure on a neighborhood V of 0 in T_eG such that V is topologically homeomorphic to a clopen subgroup U in G (see also Chapter 3 and the papers about construction of quasi-invariant measures on the considered here groups [Lud96, Lud99t, Lud98s, Lud00a, Lud02b]).

14. Let (G,τ_G) and (H,τ_H) be a pair of topological non-locally compact groups G, H (Banach-Lie, Frechet-Lie or groups of diffeomorphisms or loop groups) with uniformities τ_G, τ_H such that H is dense in (G,τ_G) and there is a probability measure $\mu \in M_l(G,H)$ with continuous $\rho_\mu(z,g)$ on $H \times G$. Also let X be a Banach space over $\mathbf{F}$ and $IS(X)$ be the group of isometric $\mathbf{F}$-linear automorphisms of X in the topology inherited from the Banach space $L(X)$ of all bounded $\mathbf{F}$ linear operators from X into X.

Theorem. (1). *If $T : G \to IS(X)$ is a weakly continuous representation, then there exists $T' : G \to IS(X)$ equal μ-a.e. to T and $T'|_{(H,\tau_H)}$ is strongly continuous.*

(2). *If $T : G \to IS(X)$ is a weakly measurable representation and X is of separable type $c_0(\mathbf{F})$ over $\mathbf{F}$, then there exists $T' : G \to IS(X)$ equal to T μ-a.e. and $T'|_{(H,\tau_H)}$ is strongly continuous.*

Proof. Take $\mathcal{K}(G) := (I) \cup L(G,\mu,\mathbf{F})$, where I is the unit operator on $L(G,\mu,\mathbf{F})$. Then we can define

$$A_{(\lambda e+a)_h} := \lambda I + \int_G a_h(g)[\rho_\mu(h,g)]T_g\mu(dg),$$

where $a_h(g) := a(h^{-1}g)$. Then

$$|(A_{(\lambda e+a)_h} - A_{\lambda e+a}\xi,\eta)| \le \sup_{g\in G}[|a_h(g)\rho_\mu(h,g) - a(g)|\ |T_g\xi|N_\mu(g)],$$

hence A_{a_h} is strongly continuous with respect to $h \in H$, that is,

$$\lim_{h\to e}|A_{a_h}\xi - A_a\xi| = 0.$$

Denote $A_{a_h} = T'_h A_a$, so $T'_h\xi = A_{a_h}\xi_0$, where $\xi = A_a\xi_0$, $a \in L(G,\mu,\mathbf{F})$. Whence

$$T'_h\xi = A_{a_h}\xi_0 = \int_G a_h(g)T_g\xi_0\rho_\mu(h,g)\mu(dg)$$

$$= T_h\int_G a(z)T_z\xi_0\mu(dz) = T_h\xi,$$

hence $|T'_h\xi| = |\xi|$ for each $h \in H$. Therefore, T'_h is uniquely extended to an isometric operator on the Banach space $X' \subset X$. In view of Lemma 10, $\|H\|_\mu = 0$. Hence T' may be considered equal to T μ-a.e. Then a space $span_{\mathbf{F}}[A_{a_h} : h \in H]$ is evidently dense in X, since

$$A_{a_h}\xi = \int_G a_h(g)T_g\rho_\mu(h,g)\mu(dg)\xi$$

$$= T_h\int_G a(g)T_g\mu(dg)\xi.$$

For proving the second statement let
$\mathsf{R} := [\xi : A_a\xi = 0$ for each $a \in L(G,\mu,\mathbf{F})]$. If $\eta \in X^*$ and

$$\eta(A_a\xi) = \eta\left(\int_G a(g)T_g\xi\mu(dg)\right) = \eta\left(\int_G a(g)T'_g\xi\mu(dg)\right)$$

for each $a(g) \in L(G,\mu,\mathbf{F})$, then $\eta(T_g\xi) = \eta(T'_g\xi)$ for μ-almost all $g \in G$, where X^* denotes the topological dual space of all $\mathbf{K}$-linear functionals $f : X \to \mathbf{K}$. Suppose that $\{\xi^*_n : n \in \mathbf{N}\}$ is an orthonormal system in X^* separating points of X. It exists, since by the supposition of this theorem $X = c_0(\mathbf{F})$. If $\xi \in X$, then

$$\xi^*_m\left(\int_G a(g)T_g\xi\mu(dg)\right) = 0$$

for each $g \in G\setminus S_m$, where $\|S_m\|_\mu = 0$. Therefore, $\xi^*_m(T_g\xi) = 0$ for each $m \in \mathbf{N}$, if $g \in G\setminus S$, where $S := \bigcup_{m=1}^\infty S_m$. Hence $T_g\xi = 0$ for each $g \in G\setminus S$, consequently, $\xi = 0$. Consider the embedding $X \hookrightarrow X^*$ with the help of the standard orthonormal basis $\{e_j : j\}$ in X over $\mathbf{F}$. Therefore, let $\{\xi_m : m \in \mathbf{N}\} \subset X \hookrightarrow X^*$. Then $\xi^*_m(T_g\xi_n) = \xi^*_m(T'_g\xi_n)$ for each $g \in G\setminus\gamma_{n,m}$, where $\|\gamma_{n,m}\|_\mu = 0$, ξ^*_m is the image of ξ_m under this standard embedding $X \hookrightarrow X^*$. Hence

$\xi_m^*(T_g\xi_n) = \xi_m^*(T'_g\xi_n)$ for each $n,m \in \mathbf{N}$ and each $g \in G \setminus \gamma$, where $\gamma := \bigcup_{n,m} \gamma_{n,m}$ and $\|\gamma\|_\mu = 0$, since in $L(G,\mu,\mathbf{F})$ the family of all step functions is dense. Therefore, we get $\mathsf{R} = 0$.

15. Definition and note. Let $\{G_i : i \in \mathbf{N_o}\}$ be a sequence of topological groups such that $G = G_0$, $G_{i+1} \subset G_i$ and G_{i+1} is dense in G_i for each $i \in \mathbf{N_o}$ and their topologies are denoted τ_i, $\tau_i|_{G_{i+1}} \subset \tau_{i+1}$ for each i, where $N_o := \{0,1,2,\dots\}$. Suppose that these groups are supplied with $\mathbf{F}$-valued probability quasi-invariant measures μ^i on G_i relative to G_{i+1}. For example, such sequences exist for groups of diffeomorphisms or wrap (particularly loop) groups considered in previous papers [Lud96, Lud99t, Lud98s, Lud00a, Lud02b, Lud08]). Let $L_{G_{i+1}}(G_i,\mu^i,\mathbf{F})$ denotes the Banach subspace of $L(G_i,\mu^i,\mathbf{F})$ as in § 1(b). Let $\tilde{L}(G_{i+1},\mu^{i+1},L(G_i,\mu^i,\mathbf{F})) =: H_i$ denotes the completion of the subspace of $L(G_i,\mu^i,\mathbf{F})$ of all elements f such that

$$\|f\|_i := \max[\|f^2\|^{1/2}{}_{L(G_i,\mu^i,\mathbf{F})}, \|f\|'_i] < \infty, \text{ where}$$

$$\|f\|'_i := [\sup_{x\in G_i, y\in G_{i+1}} |f(y^{-1}x)|^2 N_{\mu^i}(x)\max(1,N_{\mu^{i+1}}(y))]^{1/2}.$$

Evidently H_i are Banach spaces over $\mathbf{F}$. Let

$$f^{i+1} * f^i(x) := \int_{G_{i+1}} f^{i+1}(y) f^i(y^{-1}x)\mu^{i+1}(dy)$$

denotes the convolution of $f^i \in H_i$.

16. Lemma. *The convolution* $* : H_{i+1} \times H_i \to H_i$ *is the continuous* $\mathbf{F}$*-bilinear mapping.*

Proof. From the definitions we have:

$$\|(f^{i+1} * f^i)^2\|^{1/2}_{L(G_i,\mu^i,\mathbf{F})} = \sup_{x\in G_i} |(f^{i+1} * f^i)(x)| N^{1/2}_{\mu^i}(x)$$

$$\leq \sup_{x\in G_i, y\in G_{i+1}} [|f^{i+1}(y)| N^{1/2}_{\mu^{i+1}}(y)][|f^i(y^{-1}x)| N^{1/2}_{\mu^i}(x) N^{1/2}_{\mu^{i+1}}(y)]$$

$$= \|(f^{i+1})^2\|^{1/2}_{L(G_{i+1},\mu^{i+1},\mathbf{F})} \|f^i\|'_i \text{ and}$$

$$\|f^{i+1} * f^i\|'_i = [\sup_{x\in G_i, y\in G_{i+1}} |(f^{i+1} * f^i)(y^{-1}x)|^2 N_{\mu^i}(x) N_{\mu^{i+1}}(y)]^{1/2}$$

$$\leq [\sup_{x\in G_i, y\in G_{i+1}, z\in G_{i+1}} |f^{i+1}(z)|^2 N_{\mu^{i+1}}(z) |f^i(y^{-1}z^{-1}x)|^2 N_{\mu^i}(x) N_{\mu^{i+1}}(y) N_{\mu^{i+1}}(z)]^{1/2}$$

$$\leq \|(f^{i+1})^2\|^{1/2}_{L(G_{i+1},\mu^{i+1},\mathbf{F})} \|f^i\|'_i,$$

since from $A \ni y$ and $B \ni z$ for $A,B \in Af(G_{i+1},\mu^{i+1})$ it follows that $AB \ni yz$ and $AB \in Af(G_{i+1},\mu^{i+1})$, which follows from

$$\mu^{i+1}(AB) = \int_{A\ni a} \mu^{i+1}(aB)\mu^{i+1}(da) = \int_{B\ni b}\int_{A\ni a} \mu^{i+1}(adb)\mu^{i+1}(da),$$

so that $\|A\|_{\mu^{i+1}} \leq \|G_{i+1}\|_{\mu^{i+1}} = 1$, hence $N_{\mu^{i+1}}(z) \leq 1$ for each $z \in G_{i+1}$ for the probability measure μ^{i+1}. Therefore, $\|f^{i+1} * f^i\|_i \leq \|f^{i+1}\|_{i+1}\|f^i\|_i$ and inevitably the convolution $* : H_{i+1} \times H_i \to H_i$ is the continuous $\mathbf{F}$-bilinear mapping.

17. Definition. Let $c_0(\{H_i : i \in \mathbf{N_o}\}) =: H$ be the Banach space consisting of elements $f = (f^i : f^i \in H_i, i \in \mathbf{N_o})$, for which $\lim_{i\to\infty} \|f^i\|_i = 0$, where

$$\|f\| := \sup_{i=0}^{\infty} \|f^i\|_i < \infty.$$

For elements f and $g \in H$ their convolution is defined by the formula: $f \star g := h$ with $h^i := f^{i+1} * g^i$ for each $i \in \mathbf{N_o}$. Let $* : H \to H$ be an involution such that $f^* := (f^{j\wedge} : j \in \mathbf{N_o})$, where $f^{j\wedge}(y_j) := f^j(y_j^{-1})$ for each $y_j \in G_j$, $f := (f^j : j \in \mathbf{N_o})$.

18. Lemma. *H is a non-associative non-commutative Banach algebra with involution $*$, that is $*$ is $\mathbf{F}$-bilinear and $f^{**} = f$ for each $f \in H$.*

Proof. In view of Lemma 16 the convolution $h = f \star g$ in the Banach space H has the norm $\|h\| \le \|f\|\ \|g\|$, hence is a continuous mapping from $H \times H$ into H. From its definition it follows that the convolution is $\mathbf{F}$-bilinear. It is non-associative as follows from the computation of i-th terms of $(f \star g) \star q$ and $f \star (g \star q)$, which are $(f^{i+2} * g^{i+1}) * q^i$ and $f^{i+1} * (g^{i+1} * q^i)$ respectively, where f, g and $q \in H$. It is non-commutative, since there are f and $g \in H$ for which $f^{i+1} * g^i$ are not equal to $g^{i+1} * f^i$. From $f^{j\wedge\wedge}(y_j) = f^j(y_j)$ it follows that $f^{**} = (f^*)^* = f$.

19. Note. In general $(f \star g^*)^* \ne g \star f^*$ for f and $g \in H$, since there exist f^j and g^j such that $g^{j+1} * (f^j)^* \ne (f^{j+1} * (g^j)^*)^*$. If $f \in H$ is such that $f^j|_{G_{j+1}} = f^{j+1}$, then

$$((f^{j+1})^* * f^j)(e) = \int_{G_{j+1}} (f^{j+1}(y))^2 \mu^{j+1}(dy), \text{ hence}$$

$$|((f^{j+1})^* * f^j)(e)| \le \|(f^{j+1})^2\|^{1/2}_{L(G_{j+1},\mu^{j+1},\mathbf{F})} \le \|f^{j+1}\|_{j+1},$$

where $j \in \mathbf{N_o}$.

20. Definition. Consider the standard Banach space $c_0(\mathbf{F})$ over the field $\mathbf{F}$ as a Banach algebra with the convolution $\alpha \star \beta = \gamma$ such that $\gamma^i := \alpha^{i+1}\beta^i$, where $\alpha := (\alpha^i : \alpha^i \in \mathbf{F}, i \in \mathbf{N_o})$, α, β and $\gamma \in c_0(\mathbf{F})$.

21. Note. The algebra $c_0(\mathbf{F})$ has two-sided ideals $J_i := \{\alpha \in c_0(\mathbf{F}) : \alpha^j = 0$ for each $j > i\}$, where $i \in \mathbf{N_o}$. That is, $J \star c_0(\mathbf{F}) \subset J$ and $c_0(\mathbf{F}) \star J = J$ and J is the $\mathbf{F}$-linear subspace of $c_0(\mathbf{F})$, but $J \star c_0(\mathbf{F}) \ne J$. There are also right ideals, which are not left ideals: $K_i := \{\alpha \in c_0(\mathbf{F}) : \alpha^j = 0$ for each $j = 0, \dots, i\}$, where $j \in \mathbf{N_o}$. That is, $c_0(\mathbf{F}) \star K_i = K_i$, but $K_i \star c_0(\mathbf{F}) = K_{i-1}$ for each $i \in \mathbf{N_o}$, where $K_{-1} := c_0(\mathbf{F})$. The algebra $c_0(\mathbf{F})$ is the particular case of H, when $G_j = \{e\}$ for each $j \in \mathbf{N_o}$. We consider further H for non-trivial topological groups outlined above with $G_\infty := \bigcap_{j=0}^{\infty} G_j$ dense in each G_j.

22. Theorem. *If $\mathcal{F}$ is a maximal proper left or right ideal in H, then $H/\mathcal{F}$ is isomorphic as the non-associative noncommutative algebra over $\mathbf{F}$ with $c_0(\mathbf{F})$.*

Proof. The ideal $\mathcal{F}$ is also the $\mathbf{F}$-linear subspace of H. In view of Theorem 7.12 [Roo78] recalled above in § 2.32 a function $f : G_j \to \mathbf{F}$ is μ^j-integrable if and only if it satisfies two properties: f is $Af(G,\mu^j)$-continuous and for each $\varepsilon > 0$ the set $\{x : |f(x)|N_{\mu^j}(x) \ge \varepsilon\}$ is $Af(G,\mu^j)$-compact and hence contained in $\{x : N_{\mu^j}(x) \ge \delta\}$ for some $\delta > 0$. Suppose, that there exists $j \in \mathbf{N_o}$ such that $f^j = 0$ for each $f \in \mathcal{F}$, then $f^i = 0$ for each $i \in \mathbf{N_o}$, since the space of bounded $\mathbf{F}$-valued continuous functions $C^0_b(G_\infty, \mathbf{F})$ on G_∞ is dense $H_j := \{f^j : f \in H\}$ and $C^0_b(G_\infty, \mathbf{F}) \cap F_j = \{0\}$ and $C^0_b(G_j, \mathbf{F})|_{G_{j+1}} \supset C^0_b(G_{j+1}, \mathbf{F})$.

Therefore, $\mathcal{F}_j \neq \{0\}$ for each $j \in \mathbf{N_o}$, where $\mathcal{F}_j := \{f^j : f \in \mathcal{F}\}$, consequently, $\mathbf{F} \hookrightarrow F_j$ for each $j \in \mathbf{N_o}$. Since $\mathbf{F}$ is embeddable into each F_j, then there exists the embedding of $c_0(\mathbf{F})$ into $\mathcal{F}$, where $H_j := \{f^j : f \in H\}$, $\pi_j : H \to H_j$ are the natural projections.

The subalgebra $\mathcal{F}$ is closed in H, since H is the topological algebra and $\mathcal{F}$ is the maximal proper subalgebra. The space $H_\infty := \bigcap_{j \in \mathbf{N_o}} H_j$ is dense in each H_j.

If $\mathcal{F}_i = H_i$ for some $i \in \mathbf{N_o}$, then $\mathcal{F}_j = H_j$ for each $j \in \mathbf{N_o}$, since $C^0_b(G_\infty, \mathbf{F})$ is dense in each H_j and $C^0_b(G_j, \mathbf{F})|_{G_{j+1}} \supset C^0_b(G_{j+1}, \mathbf{F})$. The ideal $\mathcal{F}$ is proper, consequently, $\mathcal{F}_j \neq H_j$ as the $\mathbf{F}$-linear subspace for each $j \in \mathbf{N_o}$, where $\mathcal{F}_j = \pi_j(\mathcal{F})$.

There exist $\mathbf{F}$-linear continuous operators from $c_0(\mathbf{F})$ into $c_0(\mathbf{F})$ such that $x \mapsto (0, \dots, 0, x^0, x^1, x^2, \dots)$ with 0 as n coordinates at the beginning, $x \mapsto (x^n, x^{n+1}, x^{n+2}, \dots)$ for $n \in \mathbf{N}$; $x \mapsto (x^{kl+\sigma_k(i)} : k \in \mathbf{N_o}, i \in (0, 1, \dots, l-1))$, where $\mathbf{N} \ni l \geq 2$, $\sigma_k \in S_l$ are elements of the symmetric group S_l of the set $(0, 1, \dots, l-1)$. Then $f \star (g \star h) + c_0(\mathbf{F})$ and $(f \star g) \star h + c_0(\mathbf{F})$ are considered as the same class, also $f \star g + c_0(\mathbf{F}) = g \star f + c_0(\mathbf{F})$ in $H/c_0(\mathbf{F})$, since $(f + c_0(\mathbf{F})) \star (g + c_0(\mathbf{F})) = f \star g + c_0(\mathbf{F})$ for each f, g and $h \in H$. Then $f \star (g \star h) + c_0(\mathbf{F})$ and $(f \star g) \star h + c_0(\mathbf{F})$ are considered as the same class for each $f, g, h \in \mathcal{F}$, also $f \star g + c_0(\mathbf{F}) = g \star f + c_0(\mathbf{F})$ in $\mathcal{F}/c_0(\mathbf{F})$, since $(f + c_0(\mathbf{F})) \star (g + c_0(\mathbf{F})) = f \star g + l_2(\mathbf{F}) \subset \mathcal{F}$ for each f and $g \in \mathcal{F}$. Therefore, the quotient algebras $H/c_0(\mathbf{F})$ and $\mathcal{F}/c_0(\mathbf{F})$ are the associative commutative Banach algebras.

From $\mu^i \in M_l(G_i, G_{i+1})$ it follows that for each open subset $W \ni e$, $W \subset G_i$ there exists a clopen subgroup $U \subset W$ such that $\mu^i(U) \neq 0$, since otherwise $\mu^i(zV) = 0$ for each $z \in G_{i+1}$ and each open $V \subset W$, hence $\mu^i(G_i) = 0$ contradicting supposition, that each μ^i is the probability measure.

Let us adjoin a unit to $H/c_0(\mathbf{F})$ and to $\mathcal{F}/c_0(\mathbf{F})$. There is satisfied the equality $Ch_{G_{i+1}} * Ch_{G_i} = Ch_{G_i}$. Let $U_{i,j}$ be a clopen subgroup in G_i, that is possible, since each G_i is ultrametrizable. Choose $U_{i,j}$ such that $U_{i,j+1} \subset U_{i,j} \cap G_{j+1}$ and $U_{i,j} \supset U_{i+1,j}$ for each i and j, $\bigcap_i U_{i,j} = e \in G_j$ for each j. Since $\mu^j(G_j) = 1$ and $\|G_j\|_{\mu^j} = 1$, then by induction $(i, j) \in \{(1,1), (1,2), \dots, (1,n), \dots; (2,1), (2,2), \dots, (2,n), \dots; \dots, (m,1), (m,2), \dots, (m,n), \dots\}$, where $m, n \in \mathbf{N}$, there exists a family $\alpha_{i,j} \in \mathbf{F}$ and $\{U_{i,j} : i, j\}$ such that $\alpha_{i,j+1} Ch_{U_{i,j+1}} * \alpha_{i,j} Ch_{U_{i,j}} = \alpha_{i,j} Ch_{U_{i,j}}$ and $0 < |\alpha_{i,j}||\mu^i(U_{i,j})| \leq 1$ for each i, j. Put $e_i := \{\alpha_{i,j} Ch_{U_{i,j}} : j \in \mathbf{N_o}\}$, then $e_i * e_i = e_i$ for each i. From the properties of $U_{i,j}$ it follows, that $span_{\mathbf{F}}\{e_i(z^{-1}g) : i \in \mathbf{N}, z \in G_\infty\}$ is dense in H, where $g := (g_j : g_j \in G_j \forall j \in \mathbf{N_o})$, $z^{-1}g = (z^{-1}g_j : j)$.

Consider the algebras $H/c_0(\mathbf{F}) =: A$ and $\mathcal{F}/c_0(\mathbf{F}) =: B$. The algebras A and B are commutative and associative. From the preceding proof it follows that $span_{\mathbf{F}}\{e_i(z^{-1}g) + c_0(\mathbf{F}) : i \in \mathbf{N}, z \in G_\infty\}$ is dense in A, each $e_i(z^{-1}g) + c_0(\mathbf{F})$ is the idempotent element in A.

Remind Van der Put's theorem about C-algebras. The following conditions on a commutative Banach algebra A over a field K are equivalent:

(α) A is a C-algebra;

(β) A^+ is a C-algebra;

(γ) the linear span of $\{e \in A : e = e^2, \|e\| \leq 1\}$ is dense in A;

(δ) every $\overline{K[a]}$ with $a \in A$ is a C-algebra;

(ε) A is the smallest closed subalgebra of A that contains $\{a \in A : \overline{K[a]}$ is a C-algebra$\}$, where if A does not contain a unit element A^+ denotes the Banach algebra obtained from A by adding the unit element as $K \oplus A$ (see also Theorem 6.12 [Roo78]). Remind that a commutative Banach algebra A is called a C-algebra if there exists a locally compact zero-dimensional Hausdorff space $\mathcal{X}$ such that A is isomorphic with $C_\infty(\mathcal{X})$. A normed algebra is an algebra A with a norm such that $\|xy\| \le \|x\|\|y\|$ for all $x, y \in A$. A Banach algebra is a complete normed algebra. As usually $K[x]$ denotes the ring of polynomials over a field K, $cl_X B = \bar{B}$ denotes the closure of a subset B in a topological space X.

Therefore, by the aforementioned theorem A is the C-algebra. By the definition this means, that there exists a locally compact zero-dimensional Hausdorff space X such that A is isomorphic with $C_\infty(X,\mathbf{F})$, where $C_\infty(X,\mathbf{F})$ is the subspace of all $f \in C_b(X,\mathbf{F})$ for which for each $\varepsilon > 0$ there exists a compact subset $X_{\varepsilon,f}$ of X with $|f(x)| < \varepsilon$ for each $x \in X \setminus X_{\varepsilon,f}$.

Recall that if A is commutative Banach algebra over K, then the spectrum of A denoted by $Sp(A)$ is the set of all non-zero algebra homomorphisms from A into a field K topologized as a subset of K^A. In accordance with Theorem 6.3 [Roo78] if X is a locally compact zero-dimensional Hausdorff space, then

(Si) every $a \in X$ induces an $\hat{a} \in Sp\, C_\infty(X)$ so that $\hat{a}(f) := f(a)$, where $f \in C_\infty(X)$;

(Sii) for every closed ring ideal $\mathcal{I}$ in $C_\infty(X)$ there exists a closed subset $Z \subset X$ such that $\mathcal{I} = \{f \in C_\infty(X) : \ f = 0 \text{ on } Z\}$. In particular, each closed ring ideal is an algebra ideal;

($Siii$) to every maximal ring (or algebra) ideal M in $C_\infty(X)$ there corresponds a unique $a \in X$ such that $M = \{f \in C_\infty : \ f(a) = 0\}$, each maximal ring (or algebra) ideal of $C_\infty(X)$ is the kernel of a unique homomorphism from $C_\infty(X)$ into K;

(Siv) the mapping $a \mapsto \hat{a}$ is a homeomorphism of X onto $Sp\, C_\infty(X)$;

(Sv) suppose that $f \in C_\infty(X)$, if X is compact, then $Sp(f) = f(X)$; otherwise $Sp(f) = f(X) \cup \{0\}$. In either case, $Sp(f) = \overline{f(X)}$ and $\|f\| = \|f\|_{sp}$, where $\|x\|_{sp} := \sup_{\phi \in Sp(A)} |\phi(x)|$ for an element $x \in A$ in a commutative Banach algebra A.

In view of this theorem we get that each maximal ideal $\mathcal{B}$ of $C_\infty(X,\mathbf{F})$ has the form $\mathcal{B} = \{f \in C_\infty(X,\mathbf{F}) : \ f(z_0) = 0\}$, where z_0 is a marked point in X. On the other hand, as it was proved above $\mathcal{F}_j \ne H_j$ for each $j \in \mathbf{N_0}$, hence there exists the following embedding $c_0(\mathbf{F}) \hookrightarrow (H/\mathcal{F})$ and this implies that $(H/\mathcal{F})/c_0(\mathbf{F})$ is isomorphic with $(H/c_0(\mathbf{F}))/(\mathcal{F}/c_0(\mathbf{F}))$. Therefore, $H/\mathcal{F}$ is isomorphic with $c_0(\mathbf{F})$.

4.3. Comments

Another methods of construction of isometrical representations of topological totally disconnected groups which may be non-locally compact with the help of quasi-invariant $\mathbf{F}$-valued measures were given in [Lud98b, Lud02b, Lud00a, Lud99t, Lud01s, Lud0348, Lud01f, LD03, Lud08] and references therein.

One may mention that the non-local compactness of groups causes a twisted algebraic structure of measure spaces. This situation can be compared with representation theory of groups in non-Archimedean linear spaces. Over infinite fields with non-trivial non-Archimedean multiplicative norms the aforementioned theorem of Gelfand and Mazur is not accomplished due to existence of transcendental extensions of such fields (see Chapter 6 in [Roo78] and references therein). It was one of the basic reasons why there was demonstrated that even infinite compact groups and even commutative may have infinite-dimensional topologically irreducible representations in Banach spaces over non-Archimedean fields, for example, the additive group $\mathbf{Z_p}$ of p-adic integer numbers [Dia79, Dia84, Dia95, R84, Roo78, RS71, RS73].

The idea is the following (it was also communicated by B. Diarra). Take an algebra of bounded operators on a Banach space X over a locally compact infinite field K, consider its closed subalgebras $A_2 \subset A_1 \subset L(X,X)$ such that A_2 is a maximal closed ideal in A_1, but the quotient algebra A_1/A_2 is isomorphic with a field which is a transcendental extension F of K. Let α be a transcendental element of F over K. Put $T_1 := \beta$, where $\beta \in \theta^{-1}(\alpha)$ is an invertible operator and $\theta : A_1 \to A_1/A_2$ is the quotient mapping. This induces a strongly continuous representation of the additive group $\mathbf{Z}$ so that $T_n = T_1^n$ for each $n \in \mathbf{Z}$. As usually $GL(X)$ denotes the group of all invertible operators on X bounded together with its inverse, $GL(X) \subset L(X,X)$. For suitable F and K and A_1 and A_2, for example, when the residue class field of F contains the finite field F_p consisting of p elements, we can choose $T : Z \to GL(X)$ so that $\lim_{n\to\infty} T_{p^n}x = x$ for p-adic integers of the form p^n, for each x in an infinite dimensional closed subspace Y of X on which A_1 acts irreducibly. Then T has a strongly continuous extension on $\mathbf{Z_p}$. For this also the exponential and logarithmic functions for F can be used relating multiplicative and additive representations. Such representation may happen to be isometrical so that $T_m \in IS(X)$, where $IS(X)$ denotes the group of linear isometries of X.

The considered representation has a topologically irreducible infinite dimensional component $T : \mathbf{Z}_p \to GL(X)$. That is $cl_X\ span_K\{T_g x :\ g \in \mathbf{Z_p}\} = Y$ for each non-zero element x from some closed linear subspace Y in X, where Y is infinite dimensional over the field K.

Mention also in relation with this that each ultra-metric space can isometrically be embedded into an infinite field with a non-trivial non-Archimedean multiplicative norm in accordance with Theorem 1.10 [Sch84].

The main feature of representations $T : G \to GL(H)$ of a group G in vector spaces H over a field K as it is well-known consists in using group algebras $A(G)$ and the linear structure of H so that one has already not only a group homomorphism, but also the second operation related with addition of vectors $T_g(ax+by) = aT_g x + bT_g y$ for all $a,b \in K$ and $x,y \in H$ and $g \in G$, where $GL(H) \subset L(H,H)$ (see [Nai68, FD88]). For infinite locally compact Hausdorff topological groups the cornerstone for decomposition of unitary representations into direct integrals of topologically irreducible representations consists in using Haar measures and Banach algebras on groups associated with such measures.

In the class of non-locally compact commutative groups there exist groups G having no any finite-dimensional topologically irreducible unitary representation $T : G \to U(H)$, but having infinite dimensional strongly continuous representations $T : G \to GL(H)$, where $GL(H)$ is the general linear group on a Banach space H and $U(H)$ is the unitary group on the complex Hilbert space H. That is invariant closed subspaces in H are all infinite-dimensional in the considered case. This means that the closure of the $\mathbf{F}$-linear span $cl\ span_{\mathbf{F}}\{T_g x :\ g \in G\}$ is infinite-dimensional over the field $\mathbf{F}$ for each non-zero vector $x \neq 0$ in the linear space H over a field $\mathbf{F}$ [B83, B87, B91].

Such groups can be constructed even as quotient groups of additive groups of topological vector spaces. In accordance with Theorem 5 [B87] if E is an infinite dimensional vector space over $\mathbf{R}$ with the topological weight of E equal to the topological weight of its topological dual space E^*, then there exists a discrete subgroup Z in E such that $G := E/Z$ has no any non-trivial continuous character. This imply that such group has not any weakly continuous finite dimensional unitary representation, since it is Abelian and any weakly continuous finite dimensional unitary representations decomposes into the direct sum of characters corresponding to one-dimensional irreducible components.

A topological group G is called bounded, if for each open neighborhood V of its unit element e there exists a natural number $n \in \mathbf{N}$ so that $V^n = G$. Particularly, if $G = E/Z$, where Z is an additive subgroup of a normed space E, then G is bounded if there exists a positive number $0 < r < \infty$ so that $E = Z + rB$, where B denotes the unit ball with center at zero in E. By Proposition 3 [B87] if G is a bounded commutative topological group satisfying the first axiom of countability and T is a weakly continuous representation of G in a complex Hilbert space, then T is equivalent to a unitary weakly continuous representation. Lemma 4 in [B83] states that every bounded representation of an amenable group in a Hilbert space is equivalent to a unitary representation, where a group G is called amenable if it possess an invariant left mean on the space $C^0_b(G,\mathbf{C})$ of all continuous bounded functions on G supplied with the norm $\|f\|_{C^0_b} := \sup_{x\in G}|f(x)|$.

At the same time each topological group G admits a strongly continuous representation in a suitable Banach space S, for example, $T : G \to BUC(G,\mathbf{C})$ so that $T_g \ne I$ for each $g \ne e$, by left shifts on the Banach space $BUC(G,\mathbf{C})$ of bounded uniformly continuous complex-valued functions on G [B91]. If a Banach space H can be embedded into a Hilbert space X and this representation can weakly continuously be extended into X and if G is bounded, then this representation would be equivalent to unitary. Also when the group G is a dense subgroup in G_1 and μ is a quasi-invariant non-negative non-trivial measure on G_1 relative to left shifts from G, then G has a strongly continuous infinite dimensional unitary representation as it was outlined in [Lud06, Lud08].

This theory is rather complicated and is not considered in this book, but it is worth to note, that the non-commutative non-associative structure of measure and function algebras explain in part the known differences in the linear representation theory in non-Archimedean spaces and of non-locally compact groups in complex Hilbert spaces. Moreover, one gets more differences for representations of groups in linear spaces over infinite fields with non-Archimedean multiplicative norms. Indeed, in these cases any technique related with associative Banach algebras of functions on groups already does not work, since the algebras of functions on non-locally compact groups associated with quasi-invariant measures are already non-associative (see Lemmas 17 in Chapter III and 18 in Chapter IV) and the Gelfand-Mazur theorem is not valid for them. The initial algebras and quotient algebras as well in Theorems III.21 and IV.22 are non-associative and non-commutative and infinite-dimensional over the corresponding fields.

This is logical also due to the following. In accordance with the A. Weil's theorem if a Hausdorff group has a quasi-invariant non-trivial measure relative to itself, then it is locally compact (see [Bou63-69, FD88, VTC85] and Corollary 11 in Chapter III). Its analog for measures with values in non-Archimedean fields is valid as well (see Corollary 9 in Chapter IV).

Recall that a commutative Hausdorff topological group is called exotic if does not admit any non-trivial strongly continuous unitary representation; and strongly exotic, if it does not admit any non-trivial weakly continuous representation (not necessarily unitary) in Hilbert spaces. Using technique of exotic groups it is possible to construct further examples of Banach-Lie commutative groups having infinite dimensional strongly continuous topologically irreducible unitary representations. For wrap (particularly loop) groups and groups of diffeomorphisms they were considered in [Lud06] besides cited above works.

At first we mention the following well-known fact.

C.1. Lemma. *For an infinite dimensional complex Hilbert space X the entire unitary group $U(X)$ is not topological group relative to the weak topology τ_w.*

Proof. It is sufficient to prove this lemma for the separable complex Hilbert space $l_2(\mathbf{C})$, since $l_2(\mathbf{C})$ has the embedding into X and inevitably $U(l_2(\mathbf{C}))$ has an embedding into $U(X)$.

The weak operator topology τ_w is generated by base of neighborhoods of zero $W_b(x_1,\dots,x_n;y_1,\dots,y_n) := \{S \in L(X,X) : \ |(Sx_j,y_j)| < b \ \forall j = 1,\dots,n\}$ with $0 < b < \infty$, $n \in \mathbf{N}$, where $L(X,X)$ denotes the space of all continuous linear operators from X into X; while $(*,*)$ denotes the scalar product in X, $x_1,\dots,y_n \in X$. If $(U(X),\tau_w)$ would be a topological group, then due to continuity of the multiplication $(T,S) \mapsto TS$ and inversion $T \mapsto T^{-1}$ and the conditions $T^*T = TT^* = I$ for each $T \in U(X)$ it would be closed in (B,τ_w), where $B := B(L(X,X),0,1)$ denotes the unit ball in $L(X) := L(X,X)$ relative to the operator norm topology τ_n.

Due to the Alaoglu-Bourbaki theorem (B,τ_w) is compact [NB85]. Suppose that U_n is a Cauchy net in $U(X)$. In view of compactness of B there exists the limit $\lim_n U_n =: A$ relative to the weak topology so that $A \in B$. If $(U(X),\tau_w)$ is the topological group, then there exists $\lim_n U_n^* U_n = \lim_n U_n^* \lim_n U_n = A^*A = I$, since $U_n^*U_n = I$ for each n and the multiplication is continuous. Analogously $\lim_n U_n U_n^* = \lim_n U_n \lim_n U_n^* = AA^* = I$. Thus this would imply that $(U(X),\tau_w)$ is complete and hence closed in B. Therefore, $(U(X),\tau_w)$ would be also compact. As the compact topological group it would have a non-trivial non-negative Haar measure λ on $Bf(U(X),\tau_w)$ [Bou63-69, FD88, Nai68]. Since X is separable, then $Bf(U(X),\tau_w) = Bf(U(X),\tau_n)$.

But $U(X)$ is the Banach C^∞-manifold as well [Kl82], hence its tangent space $T_eU(X)$ is the separable Banach space, since $(U(X),\tau_n)$ is separable, where τ_n is inherited from $L(X)$. Each element in $U(X)$ lies on some one-parameter subgroup [Bou76, Kl82, FD88, RS72]. To each local one-parameter subgroup some vector in $Y := T_eU(X)$ corresponds.

Consider the exponential mapping from an open neighborhood V of zero in the algebra Y onto an open neighborhood U of the identity element e in $U(X)$. Then the linear term of $\exp(tv)$ induces the shift $w \mapsto w + tv$ in Y, where $v, tv \in V$, $w \in Y$, $t \in \mathbf{R}$. Consider embeddings $U(\mathbf{C}^n)$ into $U(X)$ for each $n \in \mathbf{N}$ and the multiplication of basic generators in $T_eU(\mathbf{C}^n)$ for each n. The multiplications in $U(X)$ and in Y are related by the Campbell-Hausdorff formula locally on sufficiently small neighborhoods U and V [Bou76]. Consider the operator $L_hg := hg$ in $U(X)$, $h,g \in U(X)$. It induces the operator L_h in Y. The multiplication in Y is $[v,w] = ad\ v(w)$. But generally operators neither $DL_h - I$ nor $(ad\ v) - I$ are compact in Y, $v \in V$, where I denotes the unit operator in Y. Nevertheless their compactness is the necessary condition for a quasi-invariance of a non trivial measure relative to L_h or $ad\ v$ in Y (see Chapter I).

Over $\mathbf{R}$ the space Y is isomorphic with $l_2(\mathbf{R})$. Thus a measure on $U(X)$ induces a measure ν on a neighborhood $V = -V$ of zero in its tangent space Y so that $0 < \nu(V) < \infty$. Since Y is separable then a measure on V induces the measure $\mu(A) := \sum_{j=1}^{\infty} \nu(A \cap (V + x_j))/2^j$ for each Borel subset A in Y, where $\{x_j : j\}$ is a set of vectors in Y so that $x_1 = 0$ and $\bigcup_{j=1}^{\infty}(x_j + V) = Y$. From the left invariance of λ relative to the entire $U(X)$ it would follow that μ is quasi-invariant on Y relative to Y, but it can be quasi-invariant relative to neither L_h nor $ad\ v$, where $h \in U$, $v \in V$. But this is impossible due to Theorem I.3.18 and Corollary I.3.19. This contradiction finishes the proof.

This can also be demonstrated directly using operators. For non-Archimedean Banach spaces an analogous result is valid.

C.2. Lemma. *For an infinite dimensional Banach space X over infinite locally compact field F with a non-Archimedean non-trivial multiplicative norm the entire isometry group $IS(X)$ is not topological group relative to the weak topology τ_w.*

Proof. It is sufficient to prove this lemma for X of separable type over F, since if $X \hookrightarrow X_1$ is the embedding of Banach spaces, then there exists the embedding $IS(X) \hookrightarrow IS(X_1)$ of groups.

For the Banach space X the topologically dual space X^* separates points in X, since F is locally compact and so spherically complete [NB85, Roo78]. A base of neighborhoods of the weak topology τ_w in $L(X,X)$ is: $W(b;x_1,\dots,x_n;y_1,\dots,y_n) := \{A \in L(X,X) : |y_j(Sx_j)| < b\}$, where $0 < b < \infty$, $x_1,\dots,x_n \in X$, $y_1,\dots,y_n \in X^*$, $n \in \mathbf{N}$. Each isometric operator S is characterized by the condition: $\|Sx\|_X = \|x\|_X$ for each $x \in X$.

Therefore, if $(IS(X),\tau_w)$ would be a topological group, then we shall show that it would be closed in (B,τ_w), where $B := B(L(X,X),0,1)$ is the unit ball in $L(X,X)$ relative to the operator norm. By the Alaogly-Bourbaki theorem [NB85] (B,τ_w) is compact for the locally compact field F.

Consider a Cauchy net U_n in $(IS(X),\tau_w)$. Since (B,τ_w) is compact, then there exists $\lim_n U_n = A \in B$ relative to the weak topology. In view of the Hahn-Banach theorem (8.4.7) [NB85] for each $x \in X$ there exists $y \in X^*$ in the topological dual space X^* so that $|y(x)| = |x|$. Thus for each $x \in X$ and each U_n there exists $y \in X^*$ so that $|y(U_n x)| = |x|$, since $|U_n x| = |x|$ for each $x \in X$ and $U_n \in IS(X)$. Let $x \in X$ be non-zero, take $0 < \varepsilon < |x|$. For any $y \in X^*$ and each such ε and x there exists n_0 such that $|y(U_n x) - y(Ax)| < \varepsilon$ for all $n > n_0$. On the other hand, $\sup_{y \in X^*, |y|=1} |y(U_n x)| = |U_n x| = |x|$, since $|y(z)| \le |y||z|$ for each $z \in X$ and $y \in X^*$. Take particularly y such that $|y(Ax)| = |Ax|$, then $|y(U_n x) - y(Ax)| < \varepsilon$ for each $n > n_0 = n_0(x,y)$, hence $|y(U_n x)| = |Ax|$, consequently, $|Ax| = |x|$, since $A \in B \subset L(X,X)$ and $y \circ A \in X^*$ and $Ax \in X$ and also $\lim_n[\lim_m y(U_n^{-1}U_m z)] = \lim_n y(U_n^{-1}Az) = \lim_n y(AU_n^{-1}z) = y(z)$ for all $y \in X^*$ and $z \in X$. Thus $(IS(X),\tau_w)$ would be complete and hence closed in B.

Therefore, as the compact group $(IS(X),\tau_w)$ would have a Haar $\mathbf{R}$-valued on $Bf(IS(X),\tau_w) = Bf(IS(X),\tau_n)$ and also $\mathbf{K_s}$-valued measures on $Bco(IS(X)),\tau_w)$, where $s \ne p$, while the residue class field of F contains the finite field $\mathbf{F_p}$.

Then consider the exponential mapping from an open neighborhood V of zero in the algebra Y onto an open neighborhood U of the identity element e in $IS(X)$. We get that the linear term of $\exp(tv)$ induces the shift $w \mapsto w + tv$ in Y, where $v, tv \in V$, $w \in Y$, $t \in \mathbf{F}$. There exist embeddings $IS(F^n)$ into $IS(X)$ for each $n \in \mathbf{N}$ and the multiplication of basic generators in $T_e IS(F^n)$ for each n. The multiplications in $IS(X)$ and in Y are related by the Campbell-Hausdorff formula locally on sufficiently small neighborhoods U and V [Bou76]. Consider the left multiplication operator $L_h g := hg$ in $IS(X)$, $h,g \in U(X)$. It induces the operator L_h in Y. The multiplication in Y is $[v,w] = ad\ v(w)$. But generally operators neither $DL_h - I$ nor $(ad\ v) - I$ are compact in Y, $v \in V$, where I denotes the unit operator in Y. Nevertheless their compactness is the necessary condition for a quasi-invariance of a non trivial measure relative to L_h or $ad\ v$ in Y (see Chapters I and II).

The isometry group $IS(X)$ is the Banach manifold as well and its tangent space $T_e IS(X)$ is isomorphic with the Banach space of separable type over F [Bou76, Roo78]. Using Theorem II.3.13 and Corollary II.3.14 for the $\mathbf{K_s}$-valued measure or Theorem I.3.18 and

Corollary I.3.19 for the **R**-valued measure analogously to § C.1 we infer the statement of this lemma.

Nevertheless, for a complex Hilbert space X weak and strong continuity of a unitary representation $T : G \to U(X)$ of a topological group G are equivalent, but this is another thing, because it is already supposed that G is the topological group [Nai68, FD88].

Recall that a locally convex representation T of a topological group G in a locally convex space X over a field F is called topologically irreducible, if $T : G \to GL(X)$, $T_g \neq I$ for some $g \in G$, and there is not any closed F-linear subspace other than $\{0\}$ and X stable (invariant) under T, that is $T_g X \subset X$ for each $g \in G$.

C.3. Theorem. *For each infinite dimensional real Hilbert space E_2 there exists a discrete additive subgroup H_2 in E_2 such that E_2/H_2 has no any continuous character, but has infinite dimensional topologically irreducible strongly continuous unitary representations. The family of such pairwise non-equivalent representations is at least* $\mathsf{c} := card(\mathbf{R})$.

Proof. It is sufficient to demonstrate this theorem for the separable Hilbert space E. If X is a non-separable infinite dimensional Hilbert space, then $X = l_2 \oplus Y$, where $Y = X \ominus l_2$ and there is the natural projection operator $P : X \to l_2$, $P(x+y) = x$ for each $x \in l_2$ and $y \in Y$. If X and l_2 are considered as additive groups this gives the homomorphism.

In accordance with [B91] there exists a discrete subgroup H in X so that X/H is exotic and $H = H_1 \oplus H_2$, where $H_1 = H \cap l_2$ and $H_2 = Y \cap H$ up to an isomorphism of Hilbert spaces. Then $P(H) = H_1$ and P induces the quotient mapping from X/H onto l_2/H_1, so that l_2/H_1 is also exotic.

Remind that a symmetric Hilbert-Schmidt non-degenerate operator A in the separable real Hilbert space E_2 has an orthonormal system of vectors $\{f_j : j \in \mathbf{N}\}$ and numbers $s_j \in \mathbf{R}$ such that $Af = \sum_j s_j (f, f_j) f_j$ for each $f \in E_2$, where $\sum_j s_j^2 < \infty$. For the nuclear (trace) operator $\sum_j |s_j| < \infty$. If A is positive definite, then $s_j > 0$ for each j. If λ_j are eigenvalues of such positive symmetric nuclear operator $A = A^* > 0$, then $0 < \sum_j \lambda_j \leq \sum_j s_j$. The product of two Hilbert-Schmidt operators is the nuclear operator [Pie65].

Let now E be separable and H be its discrete subgroup so that E/H is the exotic group (see Theorems 5.3 and 6.1 [B91]). The procedure of a choice of H described there is inductive related with a consideration of volumes of some convex bodies in linear subspaces in E, so that each generator v_j of H on j-th step is chosen from some suitable open subset in the Euclidean subspace $\mathbf{R}^{\mathbf{n(j)}}$ embedded into E.

Construct the rigged Hilbert space $E_2 \hookrightarrow E_1 \hookrightarrow E$ with linear embeddings $\theta_1 : E_1 \to E$ and $\theta_2 : E_2 \to E_1$ such that $A_1^{-1} : E \to E_1$ and $A_2^{-1} : E_1 \to E_2$, where A_1 and A_2 are symmetric non-degenerate positive definite operators of Hilbert-Schmidt class, hence $A = A_1 A_2$ is the trace class (nuclear) operator, choose these operators satisfying $A_2 = A_1|_{E_2}$ with $\|A_2\|_{L(E_2,E_1)} \leq 1$ and $\|A_1\|_{L(E_1,E)} \leq 1$ [DF91]. The scalar products in these spaces are such that $(x,y)_{E_1} = (A_1^{-1}x, A_1^{-1}y)_E$ for each $x,y \in E_1$ and $(x,y)_{E_2} = (A_2^{-1}x, A_2^{-1}y)_{E_1} = (A^{-1}x, A^{-1}y)_E$ for all $x,y \in E_2$, where $(*,*)_{E_2}$, $(*,*)_{E_1}$ and $(*,*)_E$ denote scalar products in these Hilbert spaces E_2, E_1, E (see § § II.2.1 and 2.2 in [DF91]). Therefore, $\|\theta(x)\|_E \leq \|\theta_2(x)\|_{E_1} \leq \|x\|_{E_2}$ for each $x \in E_2$, where $\theta(x) = \theta_1(\theta_2(x))$ for all $x \in E_2$. Choose H contained in E_2, that is possible since H is discrete relative to $\| * \|_E$, hence H is discrete in E_1 and E_2 relative to their norms. Therefore, θ_j induce embeddings $\theta_1 : E_1/H_1 \to E/H$ and $\theta_2 : E_2/H_2 \to E_1/H_1$ and their composition $\theta : E_2/H_2 \to E/H$, where $H_1 := \theta_2(H_2)$, $H = \theta_1(H_1) = \theta(H_2)$.

The cylinder Gaussian distribution on E_1 with the unit correlation operator and zero mean value (centered) induces the Gaussian σ-additive probability measure ν on E quasi-invariant relative to E_1. The correlation operator of such Gaussian measure is A_1. This induces the measure μ on E/H quasi-invariant relative to E_1/H_1 for suitable correlation operator A_1. If Q is a ν_y-null subset in E with a marked $y \in E_1$, then $\nu_z(Q+H) = 0$ for each $z \in E_1$, since ν is countably additive and quasi-invariant relative to shifts on vectors from E_1. For each Borel subset Q in E/H its counter-image $q^{-1}(Q)$ is a Borel subset in E so that $\nu(q^{-1}(Q)) = \mu(Q)$, where $q : E \to E/H$ is the quotient mapping. Thus, $\mu_a \ll \mu_b$ for all $a,b \in E_1/H_1$ and hence $\mu_a \sim \mu$ for each $a \in E_1/H_1$, since E_1/H_1 is the additive dense subgroup embedded into E/H.

Consider an arbitrary open subset V in E such that $V \cap H \subset \{0\}$, which is possible, since H is discrete in E. Then in E/H its image is $V+H$. Therefore,

$$\nu(V+H) = \sum_{h\in H} \nu(V+h) \text{ and} \tag{1}$$

$$\begin{aligned} \mu_a(dx)/\mu(dx) &= [\mu_a(dx)/\nu(dy)]/[\mu(dx)/\nu(dy)] \\ &= \left[\sum_{h\in H} \nu_{b+h}(dy)/\nu(dy)\right] \Big/ \left[\sum_{h\in H} \nu_h(dy)/\nu(dy)\right] \\ &= \left[\sum_{h\in H} \rho_\nu(h+b,y)\right] \Big/ \left[\sum_{h\in H} \rho_\nu(h,y)\right], \end{aligned} \tag{2}$$

where $a = b+H$, $b \in E_2$, $x = y+H$, b and y are vectors in E_2 of minimal absolute values $\|*\|_{E_2}$ satisfying these conditions. In accordance with Theorem I.4.2 [DF91]

$$\rho_\nu(h+b,y) = \exp\{(A_1^{-1}(h+b),y) - (A_1^{-1/2}(h+b), A_1^{-1/2}(h+b))/2\},$$

where $(*,*) = (*,*)_E$ is the scalar product in E, hence series in (2) are converging and continuous on $G \times (E/H)$, where $G := E_2/H_2$ (see Remark I.4.1 [DF91]). Indeed, the function

$$\omega(b,y) := \sum_{h\in H} \exp\{(A_1^{-1}(h+b),y) - (A_1^{-1/2}(h+b), A_1^{-1/2}(h+b))/2\}$$

is positive and continuous on $E_2 \times E$ due to the Weierstrass theorem and estimates of the remainder due to the generalization of Cauchy integral theorem for series of positive addends and using σ-additive Gaussian measures on E_2 up to positive multipliers. Moreover, the latter function $\omega(b,y)$ on product of bounded balls in $E_2 \times E$ is bounded from below, since $\inf_{h\in H\setminus 0} \|h\| > 0$.

Consider independent generators $\{v_j : \; j \in \mathbf{N}\}$ of H_2 so that each $h \in H_2$ is the linear combination of v_j with integer expansion coefficients $m_j \in \mathbf{Z}$, where $v_j \neq 0$ for each j.

Suppose that for A_2 eigenvectors w_j are given with eigenvalues σ_j^2, $A_2 w_j = \sigma_j^2 w_j$. Certainly $cl_{E_2} span_{\mathbf{C}}\{w_j : \; j\} = E_2$. Then

$$\rho_\nu(h+b,y) = \lim_{n\to\infty} \exp\left\{\sum_{k=1}^n [(y_k(h_k+b_k)/\sigma_k^2) - (h_k+b_k)^2/(2\sigma_k^2)]\right\}, \tag{3}$$

where y_k denotes coordinates of y in the basis $v_k/\|v_k\|_E$, $h \in H_2$, $b \in E_2$.

Consider strongly continuous unitary regular representation of $G := E_2/H_2$ in the Hilbert space $L^2(E/H,\mu,\mathbf{C}) =: X$ associated with the quasi-invariant measure μ (see [Lud02b, Lud06, Lud08]). Remind its construction. The scalar product in X is

$(f,g) := \int_{E/H} f(y)\bar{g}(y)\mu(dy)$,

where $\bar{z}$ denotes the complex conjugated number z, $f : E/H \to \mathbf{C}$. Take a marked number $b \in \mathbf{R}$, $i = (-1)^{1/2}$, and put

$$T_h f(y) := \rho_\mu^{ib+1/2}(h,y) f(h^{-1}y)$$

for each $h \in G$ and $f \in X$, where $\rho_\mu(h,y) = \mu_h(dy)/\mu(dy)$, $\mu_h(A) := \mu(h^{-1}A)$ for each Borel subset A in E/H and every $h \in G$.

Suppose that $T : G \to U(X)$ has a finite dimensional reducible component. Each finite dimensional unitary representation decomposes into direct sum of characters up to an intertwining operator on X. We demonstrate that each continuous character $\psi : G \to S^1$, where S^1 is the unit circle in $\mathbf{C}$, gives the continuous character $\chi := \psi \circ \theta^{-1}$ on E/H. Indeed, to each non-trivial continuous characters ψ and also χ there correspond continuous linear functionals f_2 and f on E_2 and E respectively so that $f|_{E_2} = f_2$ is continuous, since the topology of E_2 is stronger than that of E (see [Eng86] and Proposition 4.5 in [B91]).

For each linear continuous functional f_2 on a Hilbert space E_2 there exists a decomposition $E_2 = L_1 \oplus M_2$, where L_1 is a one-dimensional $\mathbf{R}$-linear subspace in E_2 and $M_2 = ker(f_2) = f_2^{-1}(0)$ (see § III.1.6 [KF89]). Since E_2 is the subspace in E, then $L_1 \subset E$. Take the orthogonal complement M of $\theta(L_1)$ in E so that $E = L_1 \oplus M$. Put $f(x) := f_2(x_1)$, where $x = x_1 + m$ is the unique decomposition of each $x \in E$ with $x_1 \in L_1$ and $m \in M$, since E is presented as the direct sum of these subspaces L_1 and M. Therefore, f is the declared continuous extension of f_2 from E_2 onto E and inevitably there exists the continuous character χ extending the character ψ from G onto E/H.

The space E_2 is everywhere dense in E, consequently, E_2 is not contained in M. Therefore, if f_2 is non-trivial, then f is non-trivial. But E/H is exotic, hence f is trivial and inevitably f_2 is trivial, that produces the contradiction. Thus G has no any continuous character and no any weakly continuous finite dimensional unitary representation. It remains that it has the infinite dimensional strongly continuous topologically irreducible unitary representation.

By our construction E_2 is the separable Hilbert space, hence isomorphic with the standard separable Hilbert space, while H_2 is its discrete subgroup. If E_3 is not a separable Hilbert space, then there exists the quotient mapping $P : E_3/H_3 \to E_2/H_2$ induced by the projection P from E_3 onto E_2 with $P(H_3) = H_2$, where E_2 is separable and closed and embedded into E_3, while H_3 is a discrete additive subgroup in E_3, $H_2 = H_3 \cap E_2$, $E_3 = E_2 \oplus E_4$, where $E_4 = E_3 \ominus E_2$, $H_3 = H_2 + H_4$, $H_4 = H_3 \cap E_4$, $E_2 \perp E_4$. Thus any infinite dimensional topologically irreducible unitary representation T of G induces the claimed representation $T \circ P$ for E_3/H_3.

Two unitary representations associated with quasi-invariant measures are equivalent if and only if the corresponding measures are equivalent (see [Lud06, Lud08]). The family of pairwise inequivalent quasi-invariant Gaussian measures on E and hence on E/H is at least c due to Theorems about equivalence and orthogonality of Gaussian measures [DF91, Sko74]. Thus G is the desired additive group.

Mention that the topology $\tau_{E/H}$ of E/H induces the topology τ' in G strictly weaker, than the initial topology τ_G of G and the quasi-invariance factor $\rho_\mu(a,x)$ does not exist for all $a \in E/H$, consequently, the regular representation $T = T^\mu$ of (G,τ_G) has not any weakly continuous extension on $(E/H,\tau_{E/H})$ (see also Lemmas III.12 and IV.13).

C.4. Corollary. *For each infinite dimensional real Hilbert space there exists a family of topologically irreducible infinite dimensional strongly continuous unitary representations. This family has at least* c *pairwise inequivalent representations.*

Proof. Take E_2 and the quotient mapping $q : E_2 \to E_2/H_2 = G$ from §C.3. Each strongly continuous infinite dimensional unitary representation $T : E_2/H_2 \to U(X)$ induces the unitary representation $T \circ q =: S : E_2 \to U(X)$. As the composition of continuous mappings it is continuous. Moreover, $S(H_2) = I$, hence if T is topologically irreducible, so S also is such.

Suppose that G is an infinite topological group with a topology τ such that a topological density of G is $d(G,\tau) \ge \aleph_0$ and its topological character is $\chi(G,\tau) \ge \aleph_0$ (see also [Eng86]). Let also μ be a quasi-invariant non-trivial regular Radon measure on G with values in either $F = \mathbf{R}$ or in the non-Archimedean field $F = \mathbf{K_s}$. Consider an everywhere dense subset J in G of the cardinality $card(J) = d(G,\tau)$. For each point $x \in J$ its base $\mathcal{U}_x$ of open neighborhoods can be chosen of cardinality $\chi(x,\tau) = \chi(G,\tau)$, since G is the topological group.

Then the density of either $X_\tau = L^r(G,\mu,\mathbf{C})$ with $1 \le r < \infty$ or $X_\tau = L(G,\mu,\mathbf{K_s})$ correspondingly is $d(X_\tau,\|*\|_{X_\tau}) = d(G,\tau)\chi(G,\tau)d(F,|*|_F)$, since finite linear combinations of characteristic functions of open subsets U_j from the family $\{\mathcal{U}_x : x \in J\}$ are dense in this Banach spaces due to Radon property of μ and its regularity and that each infinite covering of a compact subset in G by U_j has a finite sub-covering. Particularly if G is metrizable and F is locally compact, then $d(X_\tau,\|*\|_{X_\tau}) = d(G,\tau)\aleph_0^2 = d(G,\tau)$.

Thus if the topological group G is supplied with two topologies τ and ξ and measures μ and ν as above are so that $\aleph_0 \le d(G,\tau)\chi(G,\tau) < d(G,\xi)\chi(G,\xi)$ and $d(F,|*|_F) \le d(G,\tau)\chi(G,\tau)$, then $d(X_\tau,\|*\|_{X_\tau}) < d(X_\xi,\|*\|_{X_\xi})$. This implies that X_ξ and X_τ are not isomorphic Banach spaces in the aforementioned cases.

Therefore, a problem of finding topologically irreducible components of strongly continuous representations of (G,τ) in X_τ can generally not be reduced to that of (G,ξ). For example, when (G,τ) is a metrizable separable Frechet-Lie non-discrete topological group over $\mathbf{K}$, then $card(G) \ge card(\mathbf{K}) \ge \mathsf{c} := card(\mathbf{R}) > \aleph_0 := card(\mathbf{N})$, but $d(G,\tau) = \chi(G,\tau) = \aleph_0$. When $d(F,|*|_F)$ is greater, than $\aleph_0$ it is possible to take a locally compact subfield K in F and consider a K-linear Banach space X_K so that the enlargement of the field from K to F gives from X_K the Banach space X over F that to judge whether two spaces X_τ and X_ξ are linearly topologically isomorphic or not.

If now ξ is the discrete topology in G, then $d(G,\xi) = card(G) > d(G,\tau)\chi(G,\tau) = \aleph_0^2 = \aleph_0$. Moreover, topologies τ and ξ in G may generally be incomparable and a representation strongly continuous relative to one topology may be discontinuous relative to another topology.

Mention that an algebraic reduction of a representation implies that a group is considered relative to the discrete topology. Remind that a topological space Q is called locally compact, if for each $q \in Q$ there exists a neighborhood U of q such that $cl(U)$ is the compact subspace in Q. For compact and locally compact groups C^*-algebras are used, where by the definition C^*-algebras are complete. On C^*-algebras multiplicative continuous linear functionals are taken and related closed ideals are considered [FD88]. For either compact or a locally compact group G and a weakly continuous unitary representation $T : G \to U(X)$ some tricks are used. They are caused by the facts: (i) that a continuous image of a compact set is compact, (ii) each continuous bijective mapping of the compact space on a Hausdorff space is a homeomorphism, (iii) each uniformity on the compact topological space is complete (see Theorems 3.1.10, 3.1.13, 8.3.15 in [Eng86] and [FD88, HR79]).

This also is based on completeness of C^*-algebras and their ideals. Indeed, a topological group is T_0 if and only if it is completely regular (see Theorems 4.8 and 8.4 in [HR79]), but a topological space is T_1 if and only if each its point is closed in it (see § 1.5 in [Eng86]). A subset Q of a complete uniform space Y is complete if and only if Q is closed in Y in accordance with Theorem 8.3.6 [Eng86]. Therefore, if $f : A \to B$ is a quotient mapping of algebras, where B is supplied with the quotient topology, then B is T_0 or Hausdorff if and only if $f^{-1}(0)$ is closed in A. So that a quotient of A by a non-closed ideal produces a non-Hausdorff even without any separability axiom algebra.

But this technique generally is useless for non locally compact groups. Indeed, the unitary group $U(X)$ of an infinite dimensional complex Hilbert space is not closed in the algebra $L(X,X)$ relative to the weak topology τ_w in $L(X,X)$ so that the weak closure of $U(X)$ is not a group [Nai68]. For a non locally compact topological group its image $T(G)$ in $(L(X,X),\tau_w)$ need not be complete and its completion need not be a group, moreover, $(T(G),\tau_w)$ generally may be not a topological group even when T is bijective.

Using results of this book it is possible to make further investigations of non-locally compact totally disconnected topological groups, their structures and representations, measurable operators in Banach spaces over non-Archimedean fields, apply this for the development of non-Archimedean quantum mechanics and quantum field theory, quantum gravity, superstring theory and gauge theory, etc. Certainly this measure theory is helpful for studying random functions and stochastic processes in Banach spaces and topological groups.

Appendix A

Operators in Banach Spaces

Suppose $X = c_0(\omega_0, \mathbf{K})$ is a Banach space over a locally compact non-Archimedean infinite field $\mathbf{K}$ with a non-trivial normalization and I is a unit operator on X. If A is an operator on X, then in some basis of X we have an infinite matrix $(A_{i,j})_{i,j\in\mathbf{N}}$, so we can consider its transposed matrix A^t. If in some basis the following equality is satisfied $A^t = A$, then A is called symmetric.

A.1. Lemma. *Let $A : X \to X$ be a linear invertible operator with a compact operator $(A-I)$. Then there exist an orthonormal basis $(e_j : j \in \mathbf{N})$ in X, invertible linear operators $C,E,D : X \to X$ with compact operators $(C-I)$, $(E-I)$, $(D-I)$ such that $A = SCDE$, D is diagonal, C is lower triangular and E is upper triangular, S is an operator transposing a finite number of vectors from an orthonormal basis in X. Moreover, there exists $n \in \mathbf{N}$ and invertible linear operators $A',A" : X \to X$ with compact operators $(A'-I)$, $(A"-I)$ and $(A'_{i,j} - \delta_{i,j} = 0)$ for i or $j > n$, $A"$ is an isometry and there exist their determinants $det(A')det(A") = det(A)$, $|det(A")|_{\mathbf{K}} = 1$, $det(D) = det(A)$. If in addition A is symmetric, then $C^t = E$ and $S = I$.*

Proof. In view of Lemma 2.2[Sch89] for each $c > 0$ there exists the following decomposition $X = Y \oplus Z$ into $\mathbf{K}$-linear spaces such that $\|(A-I)|_Z\| < c$, where $dim_{\mathbf{K}}Y = m < \aleph_0$. In the orthonormal basis $(e_j : j)$ for which $span_{\mathbf{K}}(e_1,\dots,e_m) = Y$ for $c \le 1/p$ we get $A = A'A"$ with $(A-I)|_Z = 0$, $|A"_{i,j} - \delta_{i,j}| \le c$ for each i,j such that $(A'_{i,j} - \delta_{i,j}) = 0$ for i or $j > n$, where $n \ge m$ is chosen such that $|A_{i,j} - \delta_{i,j}| \le c^2$ for $i > n$ and $j = 1,\dots,m$, $A_{i,j} := e^*_i(Ae_j)$, e^*_i are vectors e_i considered as linear continuous functionals $e^*_i \in X^*$. Indeed, $(A_{i,j} : i \in \mathbf{N}) = Ae_j \in X$ and $\lim_{i\to\infty} A_{i,j} = 0$ for each j. From the form of $A"$ it follows that $\|A"e_j - e_j\| \le 1/p$ for each j, consequently, $\|A"x\| = \|x\|$ for each $x \in X$. Since $A" = (A')^{-1}A$, $(A-I)$ and $(A'-I)$ being compact, hence $(A"-I)$ is compact together with $(A^{-1}-I)$, $((A')^{-1}-I)$ and $((A")^{-1}-I)$. Moreover, there exists $\lim_{k\to\infty} det(A)_k = det(A)$ $= \lim_k det((A')_k(A")_k) = \lim_k det(A')_k det(A")_k = det(A')det(A")$, where $(A)_k := (A_{i,j} : i,j \le k)$. This follows from the decompositions $X = Y_k \oplus Z_k$ for $c = c(k) \to 0$ whilst $k \to \infty$. This means that for each $c(k) = p^{-k}$ there exists $n(k)$ such that $|A_{i,j} - \delta_{i,j}| < c(k)$, $|A'_{i,j} - \delta_{i,j}| < c(k)$ and $|A"_{i,j} - \delta_{i,j}| < c(k)$ for each i or $j > n(k)$, consequently, $|A\binom{1\cdots n(k)\, i_1\cdots i_q}{1\cdots n(k)\, j_1\cdots j_q} - A\binom{1\cdots n(k)}{1\cdots n(k)}\delta_{i_1,j_1}\cdots\delta_{i_q,j_q}| < c(k)$, where $A\binom{i_1\cdots i_r}{j_1\cdots j_r}$ is a minor corresponding to rows $i_1,\dots,i_r$ and columns $j_1,\dots,j_r$ for $r,q \in \mathbf{N}$. From the ultra-metric inequality it follows that $|det(A")-1| \le 1/p$, hence $|det(A")|_{\mathbf{K}} = 1$, $det(A")_k \ne 0$ for each k, $det(A')_k = det(A')_n$

for each $k \geq n$. Using the decomposition of $det(A')_n$ by the last row (analogously by the column) we get $A'_{n,j} \neq 0$ and a minor $A'\binom{1\cdots n-1}{1\cdots j-1, j+1, n} \neq 0$. Permuting the columns j and n (or rows) we get as a result a matrix $(\bar{A'})_n$ with $\bar{A'}\binom{1\cdots n-1}{1\cdots n-1} \neq 0$. Therefore, by the enumeration of the basic vectors we get $A'\binom{1\cdots k}{1\cdots k} \neq 0$ for each $k = 1, \dots, n$, since $|det(\bar{A'})_n| = |det(A')_n|$.

Therefore, there exists the orthonormal basis $(e_j : j)$ such that $A\binom{1\cdots j}{1\cdots j} \neq 0$ for each j and $\lim_j A\binom{1\cdots j}{1\cdots j} = det(A) \neq 0$. Applying to $(A)_j$ the Gaussian decomposition and using compactness of $A - I$ due to formula (44) in § II.4[Gan88], which is valid in the case of $\mathbf{K}$ also, we get $D = diag(D_j : j \in \mathbf{N})$, $D_j = A\binom{1\cdots j}{1\cdots j}/A\binom{1\cdots j-1}{1\cdots j-1}$; $C_{g,k} = A\binom{1,\dots,k-1,g}{1,\dots,k-1,k}/A\binom{1\cdots k}{1\cdots k}$; $E_{k,g} = A\binom{1,\dots,k-1,k}{1,\dots,k-1,g}/A\binom{1\cdots k}{1\cdots k}$ for $g = k+1, k+2, \dots,$ $k \in \mathbf{N}$. Therefore, $(C - I)$, $(D - I)$, $(E - I)$ are the compact operators, $C_{i,j}$, D_j, $E_{i,j} \in \mathbf{K}$ for each i, j. Particularly, for $A^t = A$ (A^t denotes the transposed matrix for A) we get $E_{k,g} = C_{g,k}$.

A.2. Notes. 1. An isometry operator S does not influence on the results given above by the measures, since $A|Z$ and $S^{-1}A$ on X have decompositions of the form CDE, where $X \ominus Z$ is a finite-dimensional subspace of X over $\mathbf{K}$, hence $X \ominus Z$ is locally compact for the locally compact field $\mathbf{K}$. Therefore, on $X \ominus Z$ there exists the Haar measure with values in $\mathbf{R}$ or $\mathbf{Q_s}$ with $s \neq p$, $\mathbf{K} \supset \mathbf{Q_p}$.

A.2.2. For compact $A - I$ we can construct the following decomposition $A = BDB^tC$, where $BB^t = I$, $CC^t = I$, D is diagonal, B, D and E are operators on $c_0(\omega_0, \mathbf{C_p})$ with $\lim_{i,j\to\infty}(B_{i,j} - \delta_{i,j}) = \lim_j(D_j - 1)$ $\lim_{i+j\to\infty}(C_{i,j} - \delta_{i,j}) = 0$. But, in general, matrix elements of B, C, D are in $\mathbf{C_p}$ and may be in $\mathbf{C_p} \setminus \mathbf{K}$, since from the secular equation $det(A - \lambda I)$ even for symmetric matrix A over $\mathbf{K}$ in general may appear $p^{1/n}$ for $n \in \mathbf{N}$, but $\mathbf{R}$ is not contained in $\mathbf{K}$ and $\mathbf{C_p}$ is not locally compact. This decomposition is not used in the present chapter for the construction of quasi-invariant measures on a Banach space X over $\mathbf{K}$, since on $\mathbf{C_p}$ there is not any non-trivial invariant measure (or even quasi-invariant relative to all shifts from $\mathbf{C_p}$) and may be $B_{i,j}$, D_j, $C_{i,j} \in \mathbf{C_p} \setminus \mathbf{K}$. Instead of it (and apart from [Sko74]) we use the decomposition given in the Lemma A.1.

A.3. Remarks. Let $\mathbf{K}$ be a non-Archimedean infinite field with a non-trivial normalization. Let X and Y be normed spaces over $\mathbf{K}$, and $F : U \to Y$ be a function, where $U \subset X$ is an open subset. The function F is called differentiable, if for each $t \in \mathbf{K}$, $x \in U$ and $h \in X$ such that $x + th \in U$ there exists $DF(x,h) := \{dF(x+th)/dt \mid t = 0\} = \lim_{t\to 0, t\neq 0}\{F(x+th) - F(x)\}/t$ and $DF(x,h)$ linear by h, that is, $DF(x,h) =: F'(x)h$, where $F'(x)$ is a bounded linear operator (derivative). Let $\Phi^1F(x;h;t) := \{F(x+th) - F(x)\}/t$ for each $t \neq 0$, $x \in U$, $x + th \in U$, $h \in X$. If this function $\Phi^1F(x;h;t)$ has a continuous extension $\bar{\Phi}^1F$ on $U \times V \times S$, where U and V are open neighborhoods of x and 0 in X, $S = B(\mathbf{K}, 0, 1)$ and $\|\bar{\Phi}^1F(x;h;t)\| := \sup\{\|\bar{\Phi}^1F(x;h;t)\|/\|h\| : x \in U, 0 \neq h \in V, t \in S\} < \infty$ and $\bar{\Phi}^1F(x;h;0) = F'(x)h$, then F is called continuously differentiable on U, the space of all such F is denoted by $C^1(U,Y)$, where $B(X,y,r) := \{z \in X : \|z - y\|_X \leq r\}$. By induction we define $\Phi^{n+1}F(x;h(1),\dots,h(n+1);t(1),\dots,t(n+1)\} := \{\Phi^nF(x + t(n+1)h(n+1);h(1),..,h(n);t(1),\dots,t(n)) - \Phi^nF(x;h(1),\dots,h(n);t(1),\dots,t(n))\}/t(n+1)$ and the space $C^n(U,Y)$ [Lud99t, Lud98b].

The family of all bijective surjective mappings of U onto itself of class C^n is called the diffeomorphism group and it is denoted by $Diff^n(U) := C^n(U,U) \cap Hom(U)$.

A.4. Theorem. *Let* $\mathbf{K}$ *be a spherically complete non-Archimedean infinite field with non-trivial normalization,* $F \in C^n(U,Y)$ *with* $n \in \mathbf{N} := \{1,2,..\}$, $F : X \to Y$, *X and Y be Ba-*

nach spaces, U be an open neighborhood of $y \in X$, $F(X) = Y$ and $[F'(y)]^{-1}$ be continuous, where $n \in \mathbf{Z}$, $0 \le n < p$ when $char(\mathbf{K}) =: p > 0$, $n \ge 0$ when $char(\mathbf{K}) = 0$. Then there exists $r > 0$ and a locally inverse operator $G = F^{-1}$: $V := B(Y, F(y), \|F'(y)\| r) \to B(X, y, r)$, moreover, $G \in C^n(V, X)$ and $G'(F(y)) = [F'(y)]^{-1}$.

Proof. Let us consider an open subset $U \subset X$, $[y,x] := [z : z = y + t(x-y),\ |t| \le 1,\ t \in \mathbf{K}] \subset U$ and $g : Y \to \mathbf{K}$ be a continuous $\mathbf{K}$-linear functional, $f(t) := g(F(y+t(x-y))$. Such family of non-trivial g separating points of X exists due to the Hahn-Banch theorem [NB85, Roo78], since $\mathbf{K}$ is spherically complete. Then there exists $f'(t) = g(F'(y+t(x-y))(x-y))$, $f \in C^n(B(\mathbf{K},0,1),\mathbf{K})$, consequently, for $F'(z) \ne 0$, $z \in [x,y]$ there exists $b > 0$ such that for $g(F'(z)) \ne 0$ we have $|g(F(x+t(x-y))) - g(F(x))| = |g(F'(z)(x-y))| \times |t|$ for each $|t| < b$. From $|g(u(z)) - g(u(x))| = |g(z) - g(x)|$ and the Hahn-Banach it follows, that u is the local isometry, where $u(z) := [F'(y)]^{-1}(F(z))$, z, $x \in B(X,y,r)$, r is chosen such that $\|(\bar{\Phi}^1 u(x,h,t)) - h\| < \|h\|/2$ for each $x \in B(X,y,r)$, $h \in B(X,0,r)$, $t \in B(\mathbf{K},0,1)$. Therefore, $F(B(X,y,r)) \subset B(Y, F(y), \|F'(y)\| r) =: S$. Applying to the function $H(z) := z - [F'(y)]^{-1}(F(z) - q)$ for $q \in S$ the fixed point theorem we get z' such that $F'(z') = q$, since $\|H(x) - H(z)\| < b \times \|x - z\|$. From the uniform continuous by (g,h) differentiability $f_{g,h}(t) := g(F(y+th)$ by t it follows continuous differentiability of G.

A.5. Theorem. *Let $f \in C^n(U,Y)$, where $n \in \mathbf{N}$, Y is a Banach space, $n \in \mathbf{Z}$, $0 \le n < p$ when $p := char(\mathbf{K}) > 0$, $n \ge 0$ when $char(\mathbf{K}) = 0$. Then for each x and $y \in U$ is accomplished the following formula: $f(x) = f(y) + \sum_{j=1}^{n-1} f^{(j)}(y)(x-y)^j/j! + R_n(x,y)(x-y)^{n-1}$, where $f^{(j)}(y)(x-y)^j = (\bar{\Phi}^j f)\ (y; x-y, \dots, x-y; 0, \dots, 0) \times j!$, $f^{(j)}(y)h^j := f^{(j)}(y)$ $(h, \dots, h)$, $f^{(j)}(y) : U \to L_j(X^{\otimes j}, Y)$, $R_n(x,y) : U^2 \to L_{n-1}(X^{\otimes(n-1)}, Y)$ with $R_n(x,y) = o(\|x-y\|)$, where $L_j(X^{\otimes j}, Y)$ is the Banach space of continuous polylinear operators from $X^{\otimes j}$ to Y, U is open in X.*

Proof. For $n = 1$ this formula follows from the definition of $\bar{\Phi}^1 f$. For $n = 2$ let us take $R_2(x,y) = \bar{\Phi}^2 f(x; y, y)$. Evidently, $C^{n-1}(U,Y) \supset C^n(U,Y)$. Let the statement be true for $n-1$, where $n \ge 3$. Then from $\bar{\Phi}^{n-1} f(y + t_n(x-y); x-y, \dots, x-y; t_1, \dots, t_{n-1}) = \bar{\Phi}^{n-1} f(y; x-y, \dots, x-y; t_1, \dots, t_{n-1}) +\ (\bar{\Phi}^1(\bar{\Phi}^{n-1} f(y; x-y, \dots, x-y; t_1, \dots, t_{n-1}))(y; x-y; t_n)) \times t_n$ and continuity of $\bar{\Phi}^n$ it follows that $R_n(x,y)(x-y)^{n-1} = \bar{\Phi}^n f(y; x-y, \dots, x-y; 0, \dots, 0, 1) - f^{(n)}(y)(x-y)^n/n! = \alpha(x-y)(x-y)^{n-1}$, where $\lim_{x \to y,\ x \ne y} \alpha(x-y)/\|x-y\| = 0$, that is, $\alpha(x-y) = o(\|x-y\|)$.

A.6. Suppose that X and Y are Banach spaces over a (complete relative to its uniformity) locally compact field $\mathbf{K}$. Let X and Y be isomorphic with the Banach spaces $c_0(\alpha, \mathbf{K})$ and $c_0(\beta, \mathbf{K})$ and there are given the standard orthonormal bases $\{e_j : j \in \alpha\}$ in X and $\{q_j : j \in \beta\}$ in Y respectively, then each $E \in L(X,Y)$ has its matrix realization $E_{j,k} := q_k^* E e_j$, where α and β are ordinals, $q_k^* \in Y^*$ is a continuous $\mathbf{K}$-linear functional $q_k^* : Y \to \mathbf{K}$ corresponding to q_k under the natural embedding $Y \hookrightarrow Y^*$ associated with the chosen basis, Y^* is a topologically conjugated or dual space of $\mathbf{K}$-linear functionals on Y.

A.7. Let A be a commutative Banach algebra and A^+ denotes the Gelfand space of A, that is, $A^+ = Sp(A)$, where $Sp(A)$ in another words spectrum of A was defined in Chapter 6 [Roo78]. Let $C_\infty(A^+, \mathbf{K})$ be the same space as in [Roo78, LD02].

A.8. Definition. A commutative Banach algebra A is called a C-algebra if it is isomorphic with $C_\infty(X, \mathbf{K})$ for a locally compact Hausdorff totally disconnected topological space X, where $f+g$ and fg are defined point-wise for each $f, g \in C_\infty(X, \mathbf{K})$.

A.9. Remark. Fix a Banach space H over a non-Archimedean complete field $\mathbf{F}$, as above $L(H)$ denotes the Banach algebra of all bounded **F**-linear operators on H. If $b \in L(H)$ we write shortly $Sp(b)$ instead of $Sp_{L(H)}(b) := cl(Sp(span_{\mathbf{F}}\{b^n : n = 1,2,3,\dots\}))$ (see also [Roo78]).

It was proved in Theorem 2 in [Put68] in the case of **F** with the discrete normalization group, that each continuous **F**-linear operator $A : E \to H$ with $\|A\| \le 1$ from one Banach space E into another H has the form

$$A = U \sum_{n=0}^{\infty} \pi^n P_{n,A},$$

where $P_n := P_{n,A}$, $\{P_n : n \ge 0\}$ is a family of projections and $P_nP_m = 0$ for each $n \ne m$, $\|P_n\| \le 1$ and $P_n^2 = P_n$ for each n, U is a partially isometric operator, that is, $U|_{cl(\sum_n P_n(E))}$ is isometric, $U|_{E\ominus cl(\sum_n P_n(E))} = 0$, $ker(U) \supset ker(A)$, $Im(U) = cl(Im(A))$, $\pi \in \mathbf{F}$, $|\pi| < 1$ and π is the generator of the normalization group of **F**.

We restrict our attention to the case of the locally compact field **F**, consequently, **F** has the discrete valuation group. If $\|A\| > 1$ we get

$$(i) \quad A = \lambda_A U \sum_{n=0}^{\infty} \pi^n P_{n,A},$$

where $\lambda_A \in \mathbf{F}$ and $|\lambda_A| = \|A\|$. In view of [LD02] this is the particular case of the spectral integration on the discrete topological space X. Evidently, for each $1 \le r < \infty$ there exists $J \in L(H)$ for which

$$(ii) \quad \left\{ \sum_{n\ge 0} s_n^r dim_{\mathbf{F}} P_{n,J}(H) \right\}^{1/r} < \infty$$

for $1 \le r < \infty$, where J has the spectral decomposition given by Formula (i), $s_n := |\lambda_J||\pi|^n\|P_n\|$. Using this result it is possible to give the following definition.

A.10. Definition. Let E and H be two normed **F**-linear spaces, where **F** is an infinite spherically complete field with a nontrivial non-Archimedean normalization. The **F**-linear operator $A \in L(E,H)$ is called of class $L_q(E,H)$ if there exists $a_n \in E^*$ and $y_n \in H$ for each $n \in \mathbf{N}$ such that

$$(i) \quad \left(\sum_{n=1}^{\infty} \|a_n\|_{E^*}^q \|y_n\|_H^q \right) < \infty$$

and A has the form

$$(ii) \quad Ax = \sum_{n=1}^{\infty} a_n(x) y_n$$

for each $x \in E$, where $0 < q < \infty$. For each such A we put

$$(iii) \quad \nu_q(A) = \inf \left\{ \sum_{n=1}^{\infty} \left\| a_n \right\|_{E^*}^q \|y_n\|_H^q \right\}^{1/q},$$

where the infimum is taken by all such representations (ii) of A,

$$(iv) \quad \nu_\infty(A) := \|A\|$$

and $L_\infty(E,H) := L(E,H)$.

A.11. Proposition. *$L_q(E,H)$ is the normed* **F**-*linear space with the norm ν_q, when $1 \le q$; it is the metric space, when $0 < q < 1$.*

Proof. Let $A \in L_q(E,H)$ and $1 \le q < \infty$, since the case $q = \infty$ follows from its definition. Then A has the representation A.10.(ii). Then due to the ultra-metric inequality

$$\|Ax\|_H \le \|x\|_E \sup_{n\in\mathbf{N}}(\|a_n\|_{E^*}\|y_n\|_H) \le \|x\|_E \left(\sum_{n=1}^{\infty} \|a_n\|_{E^*}^q \|y_n\|_H^q\right)^{1/q},$$

hence $\sup_{x\ne 0} \|Ax\|_H/\|x\|_E =: \|A\| \le \nu_q(A)$.

Let now $A,S \in L_q(E,H)$, then there exists $0 < \delta < \infty$ and two representations $Ax = \sum_{n=1}^{\infty} a_n(x)y_n$ and $Sx = \sum_{m=1}^{\infty} b_m(x)z_m$ for which

$$\left(\sum_{n=1}^{\infty} \|a_n\|_{E^*}^q \|y_n\|_H^q\right)^{1/q} \le \nu_q(A) + \delta \text{ and}$$

$$\left(\sum_{n=1}^{\infty} \|b_n\|_{E^*}^q \|z_n\|_H^q\right)^{1/q} \le \nu_q(S) + \delta, \text{ hence}$$

$(A+S)x = \sum_{n=1}^{\infty}(a_n(x)y_n + b_n(x)z_n)$ and

$$\nu_q(A+S) \le \left(\sum_{n=1}^{\infty} \|a_n\|^q \|y_n\|^q\right)^{1/q} + \left(\sum_{n=1}^{\infty} \|b_n\|^q \|z_n\|^q\right)^{1/q} \le \nu_q(A) + \nu_q(S) + 2\delta$$

due to the Hölder inequality. The case $0 < q < 1$ is analogous to the classical one given in [Pie65].

A.12. Proposition. *If $J \in L_q(H)$, $S \in L_r(H)$ are commuting operators, the field* **F** *is with the discrete valuation group and $1/q + 1/r = 1/v$, then $JS \in L_v(H)$, where $1 \le q,r,v \le \infty$.*

Proof. Since **F** is with the discrete valuation, then J and S have the decompositions A.9.(i). Certainly each projector $P_{n,J}$ and $P_{m,S}$ belongs to $L_1(H)$ and have the decomposition A.10.(ii). The **F**-linear span of $\bigcup_{n,m} range(P_{n,J}P_{m,S})$ is dense in H. In particular, for each $x \in range(P_{n,J}P_{m,S})$ we have $J^k S^l x = \lambda_J^k \lambda_S^l \pi^{nk+ml} P_{n,J}P_{m,S}x$. Applying § A.9 to commuting operators J^k and S^l for each $k,l \in \mathbf{N}$ and using the base of H we get projectors $P_{n,J}$ and $P_{m,S}$ which commute for each n and m, consequently, $JS = U_J U_S \lambda_J \lambda_S \sum_{n\ge 0, m\ge 0} \pi^{n+m} P_{n,J}P_{m,S}$, hence $U_{JS} = U_J U_S$, $\lambda_{JS} = \lambda_J \lambda_S$, $P_{l,JS} = \sum_{n+m=l} P_{n,J}P_{m,S}$. In view of the Hölder inequality $\nu_v(JS) = \inf(\sum_{n=0}^{\infty} s_{n,JS}^v dim_{\mathbf{F}} P_{n,JS}(H))^{1/v} \le \nu_q(J)\nu_r(S)$.

A.13. Proposition. *If E is the normed space and H is the Banach space over the field* **F** *(complete relative to its uniformity), then $L_r(E,H)$ is the Banach space such that if $J,S \in L_r(E,H)$, then*

$$\|J+S\|_r \le \|J\|_r + \|S\|_r; \quad \|bJ\|_r = |b| \; \|J\|_r \textit{ for each } b \in \mathbf{K};$$

$\|J\|_r = 0$ if and only if $J = 0$, where $1 \le r \le \infty$, $\|*\|_q := \nu_q(*)$.

Proof. In view of Proposition A.11 it remains to prove that $L_r(E,H)$ is complete, when H is complete. Let $\{T_\alpha\}$ be a Cauchy net in $L_r(E,H)$, then there exists $T \in L(E,H)$ such that $\lim_\alpha T_\alpha x = Tx$ for each $x \in E$, since $L_r(E,H) \subset L(E,H)$ and $L(E,H)$ is complete. We demonstrate that $T \in L_r(E,H)$ and T_α converges to T relative to ν_r for $1 \le r < \infty$.

Let α_k be a monotone subsequence in $\{\alpha\}$ such that $\nu_r^r(T_\alpha - T_\beta) < 2^{-k-2}$ for each $\alpha, \beta \ge \alpha_k$, where $k \in \mathbf{N}$. Since $T_{\alpha_{k+1}} - T_{\alpha_k} \in L_r(E,H)$, then $(T_{\alpha_{k+1}} - T_{\alpha_k})x = \sum_{n=1}^\infty a_{n,k}(x)y_{n,k}$ with $\sum_{n=1}^\infty \|a_{n,k}\|^r \|y_{n,k}\|^r < 2^{-k-2}$. Therefore, $(T_{\alpha_{k+p}} - T_{\alpha_k})x = \sum_{h=k}^{k+p-1} \sum_{n=1}^\infty a_{n,h}(x)y_{n,h}$ for each $p \in \mathbf{N}$, consequently, using convergence while p tends to ∞ we get $(T - T_{\alpha_k})x = \sum_{h=k}^\infty \sum_{n=1}^\infty a_{n,h}(x)y_{n,h}$. Then $\nu_r^r(T - T_{\alpha_k}) \le \sum_{h=k}^\infty \sum_{n=1}^\infty \|a_{n,h}\|^r \|y_{n,h}\|^r \le 2^{-k-1}$, hence $T - T_{\alpha_k} \in L_r(E,H)$ and inevitably $T \in L_r(E,H)$. Moreover, $\nu_r(T - T_\alpha) \le \nu_r(T - T_{\alpha_k}) + \nu_r(T_{\alpha_k} - T_\alpha) \le 2^{-(k-1)/r}2$ for each $\alpha \ge \alpha_k$.

A.14. Proposition. *Let E, H, G be normed spaces over spherically complete* $\mathbf{F}$. *If $T \in L(E,H)$ and $S \in L_r(H,G)$, then $ST \in L_r(E,G)$ and $\nu_r(ST) \le \nu_r(S)\|T\|$. If $T \in L_r(E,H)$ and $S \in L(H,G)$, then $ST \in L_r(E,G)$ and $\nu_r(ST) \le \|S\|\nu_r(T)$.*

Proof. For each $\delta > 0$ there are $b_n \in H^*$ and $z_n \in G$ such that $Sy = \sum_{n=1}^\infty b_n(y)z_n$ for each $y \in H$ and $\sum_{n=1}^\infty \|b_n\|^r \|z_n\|^r \le \nu_r^r(S) + \delta$. Therefore, $STx = \sum_{n=1}^\infty T^*b_n(x)z_n$ for each $x \in E$, hence $\nu_r(ST) \le \sum_{n=1}^\infty \|T^*b_n\|^r \|z_n\|^r \le \|T\|[\nu_r^r(S) + \delta]$, since $\|T^*b_n(x)\| = |b_n(Tx)| \le \|b_n\|\|Tx\| \le \|b_n\|\|T\|\|x\|$, where $T^* \in L(H^*, E^*)$ is the adjoint operator such that $b(Tx) =: (T^*b)(x)$ for each $b \in H^*$ and $x \in E$. The operator T^* exists due to the Hahn-Banach theorem for normed spaces over the spherically complete field $\mathbf{F}$ [NB85, Roo78].

A.15. Proposition. *If $T \in L_r(E,H)$, then $T^* \in L_r(H^*, E^*)$ and $\nu_r(T^*) \le \nu_r(T)$, where E and H are over the spherically complete field* $\mathbf{F}$.

Proof. For each $\delta > 0$ there are $a_n \in E^*$ and $y_n \in H$ such that $Tx = \sum_{n=1}^\infty a_n(x)y_n$ for each $x \in E$ and $\sum_{n=1}^\infty \|a_n\|^r \|y_n\|^r \le \nu_r^r(T) + \delta$. Since $(T^*b)(x) = b(Tx) = \sum_{n=1}^\infty a_n(x)b(y_n)$ for each $b \in H^*$ and $x \in E$, then $T^*b = \sum_{n=1}^\infty y_n^*(b)a_n$, where $y_n^*(b) := b(y_n)$, that is correct due to the Hahn-Banach theorem for E and H over the spherically complete field $\mathbf{F}$ [NB85, Roo78]. Therefore, $\nu_r^r(T^*) \le \sum_{n=1}^\infty \|y_n\|^r \|a_n\|^r \le \nu_r^r(T) + \delta$, since $\|y^*\|_{H^*} = \|y\|_H$ for each $y \in H$.

A.16. Comments. Bounded operators on non-Archimedean Banach spaces were investigated, for example, in [Gru66, Put68, Roo78, LD02, Lud0341] and references therein. In this book only some specific results in operator theory are used, so there are not referred all works on this subject.

Appendix B

Non-Archimedean Polyhedral Expansions

B.1. Ultra-uniform Spaces

B.1.1. Let us recall that by an ultra-metric space (X,ρ) is implied a set X with a metric ρ such that it satisfies the ultra-metric inequality: $\rho(x,y) \le \max(\rho(x,z); \rho(z,y))$ for each x, y and $z \in X$. A uniform space X with ultra-uniformity U is called an ultra-uniform space (X,U) such that U satisfies the following condition: $|x-z| < V'$, if $|y-z| < V'$ and $|x-y| < V$, where $V \subset V' \in U$, x, y and $z \in X$ [Eng86, Roo78]. If in X a family of pseudo-ultra-metrics $\mathbf{P}$ is given and it satisfies conditions $(UP1, UP2)$, then it induces an ultra-uniformity U due to proposition 8.1.18 [Eng86].

Let $\mathbf{L}$ be a non-Archimedean field. We say that X is a $\mathbf{L}$-Tychonoff space, if X is a T_1-space and for each $F = \bar{F} \subset X$ with $x \notin F$ there exists a continuous function $f : X \to B(\mathbf{L},0,1)$ such that $f(x) = 0$, $f(F) = \{1\}$, where $B(X,y,r) := \{z \in X : \rho(y,z) \le r\}$ for $y \in X$ and $r \ge 0$. From $Ind(\mathbf{L}) = 0$ it follows $Ind(X) = 0$ (see §6.2 and Chapter 7 in [Eng86]). Since the norm $|*|_{\mathbf{L}} : \mathbf{L} \to \tilde{\Gamma}_{\mathbf{L}}$ is continuous, then X is the Tychonoff space, where $\tilde{\Gamma}_{\mathbf{L}} := \{|x|_{\mathbf{L}} : x \in \mathbf{L}\} \subset [0,\infty)$. Vice versa if X is a Tychonoff space with $Ind(X) = 0$, then it is also $\mathbf{L}$-Tychonoff, since there exists a clopen (closed and open at the same time) neighborhood $W \ni x$ with $W \cap F = \emptyset$ and as the locally constant function f may be taken with $f(x) = 0$ and $f(W) = \{1\}$.

Let us consider spaces $C(X,\mathbf{L}) := \{f : X \to \mathbf{L} | f$ is continuous $\}$ and $C^*(X,\mathbf{L}) := \{f \in C(X,\mathbf{L}) : |f(X)|_{\mathbf{L}}$ is bounded in $\mathbf{R}\}$, then for each finite family $\{f_1,\dots,f_m\} \subset C(X,\mathbf{L})$ (or $C^*(X,\mathbf{L})$) the following pseudo-ultra-metric is defined: $\rho_{f_1,\dots,f_m}(x,y) := \max(|f_j(x) - f_j(y)|_{\mathbf{L}} : j = 1,\dots,m\}$. Families $\mathbf{P}$ or $\mathbf{P}^*$ of such $\rho_{f_1,\dots,f_m}$ induce ultra-uniformities C or C^* respectively and the initial topology on X. If a sequence $\{V_j : j = 0,1,\dots\} \subset U$ is such that $V_0 = X^2$, $pV_{j+1} \subset V_j$ for $j = 1,2,\dots$, where p is a prime number, then there exists a pseudo-ultra-metric $\rho(x,y) := 0$ for $(x,y) \in \bigcap_{j=0}^{\infty} V_j$, $\rho(x,y) = p^{-j}$ for $(x,y) \in V_j \setminus V_{j+1}$, so $V_i \subset \{(x,y) : \rho(x,y) \le p^{-i}\} \subset V_{i-1}$. Indeed, from $(x,y) \in V_i \setminus V_{i+1}$ and $(y,z) \in V_j \setminus V_{j+1}$ for $j \ge i$ it follows $(x,z) \in V_i$ and $\rho(x,z) \le p^{-i} = \rho(x,y)$. Therefore, ultra-uniform spaces may be equivalently characterized by U or $\mathbf{P}$ (see § 8.1.11 and § 8.1.14 in [Eng86]).

Henceforth, locally compact non-discrete non-Archimedean infinite fields $\mathbf{L}$ are considered. If the characteristic $char(\mathbf{L}) = 0$, then due to [Wei73] for each such $\mathbf{L}$ there exists a prime number p such that $\mathbf{L}$ is a finite algebraic extension of the field $\mathbf{Q_p}$ of p-adic numbers. If $char(\mathbf{L}) = p > 0$, then $\mathbf{L}$ is isomorphic with the field $\mathbf{F_{p^n}}(\theta)$ of the formal power series by the indeterminate θ, where $n \in \mathbf{N}$, p is the prime number, each $z \in \mathbf{L}$ has the form $z = \sum_{j=k(z)}^{+\infty} a_j \theta^j$ with $a_j \in \mathbf{F_{p^n}}$ for each j, $|\theta|_{\mathbf{L}} = p^{-1}$ up to the equivalence of the non-Archimedean normalization, $k(z) \in \mathbf{Z}$, $\mathbf{F_{p^n}}$ is the finite field consisting of p^n elements, $\mathbf{F_{p^n}} \hookrightarrow \mathbf{L}$ is the natural embedding.

For an ordinal A with its cardinality $m = card(A)$ by $c_0(\mathbf{L}, A)$ it is denoted the following Banach space with vectors $x = (x_a : a \in A, x_a \in \mathbf{L})$ of a finite norm $\|x\| := \sup_{a \in A} |x_a|_{\mathbf{L}}$ and such that for each $b > 0$ a set $\{a \in A : |x_a|_{\mathbf{L}} \geq b\}$ is finite. It has the orthonormal in the non-Archimedean sense basis $\{e_j := (\delta_{j,a} : a \in A) : j \in A\}$, where $\delta_{j,a} = 1$ for $j = a$ and $\delta_{j,a} = 0$ for each $j \neq a$ [Roo78]. If $card(A_1) = card(A_2)$ then $c_0(\mathbf{L}, A_1)$ is isomorphic with $c_0(\mathbf{L}, A_2)$. In particular $c_0(\mathbf{L}, n) = \mathbf{L^n}$ for $n \in \mathbf{N}$. Then $card(A)$ is called the dimension of $c_0(\mathbf{L}, A)$, $card(A) = dim_{\mathbf{L}} c_0(\mathbf{L}, A)$.

B.1.2. Lemma. *Let (X, ρ) be an ultra-metric space, then there exists an ultra-metric ρ' equivalent to ρ such that $\rho'(X, X) \subset \tilde{\Gamma}_{\mathbf{L}}$.*

Proof. Let $\rho'(x,y) := \sup_{b \in \tilde{\Gamma}_{\mathbf{L}}, b \leq \rho(x,y)} b$, where x and $y \in X$, either $\mathbf{L} \supset \mathbf{Q_p}$ or $\mathbf{L} = \mathbf{F_{p^n}}(\theta)$. Then ρ' is the ultra-metric such that $\rho'(x,y) \leq \rho(x,y) \leq p \times \rho'(x,y)$ for each x and $y \in X$.

B.1.3. Lemma. *Let $(X, \mathbf{P})$ be an ultra-uniform space, then there exists a family $\mathbf{P}'$ such that $\rho'(X, X) \subset \tilde{\Gamma}_{\mathbf{L}}$ for each $\rho' \in \mathbf{P}'$; $(X, \mathbf{P})$ and $(X, \mathbf{P}')$ are uniformly isomorphic, the completeness of one of them is equivalent to completeness of another.*

Proof. In view of Lemma 2 for each $\rho \in \mathbf{P}$ there exists an equivalent pseudo-ultra-metric ρ'. They form a family $\mathbf{P}'$. Evidently, the identity mapping $id : (X, \mathbf{P}) \to (X, \mathbf{P}')$ is the uniform isomorphism. The last statement follows from 8.3.20 [Eng86].

B.1.4. Theorem. *For each ultra-uniform space (X, ρ) there exist an embedding $f : X \to B(c_0(\mathbf{L}, A_X), 0, 1)$ and an uniformly continuous embedding into $c_0(\mathbf{L}, A_X)$, where $card(A_X) = w(X)$, $w(X)$ is the topological weight of X.*

Proof. In view of theorem 7.3.15 [Eng86] there exists an embedding of X into the Baire space $B(m)$, where $m = w(X) \geq \aleph_0$. In the case $w(X) < \aleph_0$ this statement is evident, since X is finite. In view of lemma 2.2 we choose in $B(m)$ an ultra-metric ρ equivalent to the initial one with values in $\tilde{\Gamma}_{\mathbf{L}}$ such that $\rho(\{x_i\}, \{y_i\}) = p^{-k}$, if $x_k \neq y_k$ and $x_i = y_i$ for $i < k$, $\rho(\{x_i\}, \{y_i\}) = 0$, if $x_i = y_i$ for all i, where $\{x_i\} \in B(m)$, $i \in \mathbf{N}$, $x_i \in D(m)$, $D(m)$ denotes the discrete space of cardinality m. Let $A = \mathbf{N} \times C$, $card(C) = m$, $\{e_{i,a} | (i,a) \in A\}$ be the orthonormal basis in $c_0(\mathbf{L}, A)$. For each $\{x_i\} \in B(m)$ we have $x_i \in D(m)$ and we can take $x_i = e_{i,a}$ for suitable $a = a(i)$, since $D(m)$ is isomorphic with $\{e_{i,a} : a \in C)$. Let $f(\{x_i\}) := \sum_{i \in \mathbf{N}, a = a(i)} p^i e_{i,a}$, consequently, $\|f(\{x_i\}) - f(\{y_i\})\|_{c_0(\mathbf{L}, A)} = \rho(\{x_i\}, \{y_i\})$.

The last statement of the theorem follows from the isometrical embedding of (X, ρ) into the corresponding free Banach space, which is isomorphic with $c_0(\mathbf{L}, A)$ (see theorem 5 in [Lud95] and Theorems 5.13 and 5.16 in [Roo78]).

For each ultra-metric $\rho \in \mathbf{P}$ of an ultra-uniform space $(X, \mathbf{P})$ there exists the equivalence relation R_ρ such that $x R_\rho y$ if and only if $\rho(x,y) = 0$. Then there exists the quotient mapping $g_\rho : X \to X_\rho := (X / R_\rho)$, where X_ρ is the ultra-metric space with the ultra-metric also denoted by ρ, $\tilde{X}$ denotes the completion of X. Then X has the uniform embedding into the limit of the inverse spectra $inv - \lim_\rho X_\rho = \tilde{X}$. Hence we have got the following.

B.1.5. Corollary. *Each ultra-uniform space $(X, \mathbf{P})$ has a topological embedding into*

$\prod_{\rho\in\mathbf{P}} B(c_0(\mathbf{L},A_\rho),0,1)$ *and an uniform embedding into* $\prod_{\rho\in\mathbf{P}} c_0(\mathbf{L},A_\rho)$ *with* $card(A_\rho) = w(X_\rho$.

B.1.6. Corollary. *For each* $j\in\mathbf{N}$ *and* $f(X)\subset c_0(\mathbf{L},A_X)$ *from Theorem* 4 *there are coverings* U_j *of the space* $f(X)$ *by disjoint clopen balls* B_l *with diameters not greater than* p^{-j} *and with* $\inf_{l\neq k} dist(B_l,B_k)>0$.

Proof follows from the consideration of the covering of $c_0(\mathbf{L},A)$ by balls $B(c_0(\mathbf{L},A),x,r)$ with $0<r\le p^{-j}$ and $x\in c_0$, since such balls are either disjoint or one of them is contained in another and $\Gamma_{\mathbf{L}}$ is discrete in $(0,\infty)$, where $\Gamma_{\mathbf{L}} := \tilde{\Gamma}_{\mathbf{L}}\setminus\{0\}$. Then $\bigcup_{q\in J} B(c_0,x_q,r_q)$ is the clopen ball in c_0 with $r\le p^{-j}$, if all balls in the family J have non-void pairwise intersections. Taking $B(c_0,x,r)\cap f(X)$ we get the statement for $f(X)$ using the transfinite sequence of the covering.

B.1.7. Note. A simplex s in $\mathbf{R^n}$ may be taken with the help of linear functionals, for example, $\{e_j : j=0,\dots,n\}$, where $e_j=(0,\dots,0,1,0,\dots,0)$ with 1 in the j-th place for $j>0$ and $e_0=e_1+\dots+e_n$, $s:=\{x\in\mathbf{R^n} : e_j(x)\in[0,1]$ for $j=0,1,\dots,n\}$. In the case of $\mathbf{L^n}$, if to take $B(\mathbf{L},0,1)$ instead of $[0,1]$, then conditions $x_j := e_j(x)\in B(\mathbf{L},0,1)$ for $j=1,\dots,n$ imply $e_0(x)=x_1+\dots+x_n\in B(\mathbf{L},0,1)$ due to the ultra-metric inequality (since $B(\mathbf{L},01,)$ is the additive group), that is $s=B(\mathbf{L},0,1)$. Moreover, its topological border is empty $Fr(s)=\emptyset$ and $Ind(Fr(s))=-1$. Let us denote by $\pi_{\mathbf{L}}$ an element from $\mathbf{L}$ such that $B(\mathbf{L},0,1^-) := \{x\in\mathbf{L} : |x|_{\mathbf{L}}<1\} = \pi_{\mathbf{L}} B(\mathbf{L},0,1)$ and $|\pi_{\mathbf{L}}|_{\mathbf{L}} = \sup_{b\in\Gamma_{\mathbf{L}},b<1} b =: b_{\mathbf{L}}$.

B.1.8. Definitions. (1). A subset P in $c_0(\mathbf{L},A)$ is called a polyhedron if it is a disjoint union of simplexes s_j, $P=\bigcup_{j\in F} s_j$, where F is a set, $s_j = B(c_0(\mathbf{L},A'),x,r) = x+\pi_{\mathbf{L}}^k B(c_0(\mathbf{L},A'),0,1)$ are the clopen balls in $c_0(\mathbf{L},A')$, $A'\subset A$, $r=b_{\mathbf{L}}^k$, $k\in\mathbf{Z}$. For each $\mathbf{L}$ we fix $\pi_{\mathbf{L}}$ and such affine transformations. The polyhedron P is called uniform if it satisfies conditions (i,ii):

(i) $\sup_{i\in F} diam(s_i)<\infty$,

(ii) $\inf_{i\neq j} dist(s_i,s_j)>0$, where $dist(s,q) := \inf_{x\in s,y\in q}\rho(x,y)$. By vertices of the simplex $s=B(c_0(\mathbf{L},A),0,1)$ we call points $x=(x_j)\in c_0(\mathbf{L},A)$ such that $x_j=0$ or $x_j=1$ for each $j\in A$, $dim_{\mathbf{L}}(s) := card(A)$. For each $E\subset A$, $E\neq A$ and a vertex e by verge of the simplex s we call a subset $e+B(c_0(\mathbf{L},E),0,1)\subset s$. For an arbitrary simplex its verges and vertices are defined with the help of affine transformation as images of verges and vertices of the unit simplex $B(c_0(\mathbf{L},A'),0,1)$.

Then in analogy with the classical case there are naturally defined notions of a simplicial complex K and his space $|K|$, also a sub-complex and a simplicial mapping. The latter has restrictions on each simplex of polyhedra that are affine mappings over the field $\mathbf{L}$ and images of vertices are vertices.

Instead of the barycentric subdivision in the classical case we introduce a p^j-subdivision of simplexes and polyhedra for $j\in\mathbf{N}$ and $\mathbf{L}\supset\mathbf{Q_p}$, that is a partition of each simplex $B(c_0(\mathbf{L},A'),x,r)$ into the disjoint union of simplexes with diameters equal to rp^{-j}. Each simplex s with $dim_{\mathbf{L}}s = card(A')$ may be considered also in $c_0(\mathbf{L},A)$, where $A'\subset A$, since there exists the isometrical embedding $c_0(\mathbf{L},A')\hookrightarrow c_0(\mathbf{L},A)$ and the projector $\pi : c_0(\mathbf{L},A)\to c_0(\mathbf{L},A')$. By a dimension of a polyhedron P we call $dim_{\mathbf{L}}P := \sup_{(s\subset P, s \text{ is a simplex })} dim_{\mathbf{L}}s$. The polyhedron P is called locally finite-dimensional if all simplexes $s\subset P$ are finite dimensional over $\mathbf{L}$, that is, $dim_{\mathbf{L}}s\in\mathbf{N}$. For a simplex $s=B(c_0(\mathbf{L},A'),x,r)$ by $\mathbf{L}$-border ∂s we call the union of all its verges q with the codimension over $\mathbf{L}$ equal to 1 in $c_0(\mathbf{L},A') =: X$, that is, $q=e+B'$, where B' are balls in

$c_0(\mathbf{L},A") =: Y$, $(A' \setminus A")$ is a singleton, $A" \subset A'$, $X \ominus Y = \mathbf{L}$, $Y \hookrightarrow X$. For the polyhedron $P = \bigcup_{j\in F} s_j$ by the **L**-border we call $\partial P = \bigcup_{j\in F} \partial s_j$, where F is the set.

(2). A continuous mapping f of a set M, $M \subset c_0(\mathbf{L},A)$ into a polyhedron P we call essential, if there is not any continuous mapping $g : M \to P$ for which are satisfied the following conditions:

(i) $g(M)$ does not contain P;

(ii) there exists $M_0 \subset M$, $M_0 \neq M$ with $f(M) \cap \partial P = f(M_0) = g(M_0) \subset \partial P$ and their restrictions coincide $f|_{M_0} = g|_{M_0}$;

(iii) if f is linear on $[x,y] := \{tx + (1-t)y | t \in B(\mathbf{L},0,1)\} \subset M$, then g is also linear on $[x,y]$ such that $g(x) \neq g(y)$ for each $f(x) \neq f(y)$.

(3). The function f from § 2.8.(2) is inessential, if there exists such g.

(4). Let $f : M \to N$ be a continuous mapping, $c_0(\mathbf{L},A) \supset N \supset P$, P be a polyhedron. Then P is called essentially (or unessentially) covered by N under the mapping f, if $f|_{f^{-1}(P)}$ is essential (or inessential respectively).

(5). Let $f : M \to P$ and $g : M \to P$ are continuous mappings, where M is a set, P is a polyhedron. Then g is called a permissible modification of f, if three conditions are satisfied:

(i) from $a \in M$ and $f(a) \in s$ it follows $g(a) \in s$, where s is a simplex from P;

(ii) if x and $y \in M$, $[x,y] \subset M$ and $f : [x,y] \to P$ is linear, then $g : [x,y] \to P$ is also linear (over **L**) and $g(x) \neq g(y)$ for each $f(x) \neq f(y)$;

(iii) $f(\partial M) = g(\partial M)$.

(6). The mapping $f : M \to P$ is called reducible (or irreducible), when it may (or not respectively) have the permissible modification g such that $f(M)$ is not the subset of $g(M)$.

(7). A mapping $f : P \to Q$ for polyhedra P and Q is called normal, if

(i) $\rho_Q(f(x), f(y)) \leq \rho_P(x,y)$ for each x and $y \in P$, that is f is a non-stretching mapping;

(ii) there exists a p^j-subdivision Q' of polyhedra Q such that $f : P \to Q'$ is a simplicial mapping, that is, $f|_s$ is affine on each simplex $s \subset P$ and $f(s)$ is a simplex from Q'.

(8). Let $X = inv - \lim_j\{X_j, f_i^j, E\}$ be an expansion of X into the limit of inverse spectra of polyhedra X_j over **L**. This expansion is called (a) irreducible, if for each open $V \subset X$ there exists a co-final subset $E_V \subset E$ such that $\{X_j, f_i^j, E_V\}$ is also the irreducible polyhedral representation of the space V, that is $f_i^j : X_j \to X_i$ are irreducible and surjective for each $i \geq j \in E_V$. The polyhedral system (representation) $\{X_j, f_i^j, E\}$ is called (b) n-dimensional, if $dim_{\mathbf{L}} X_j \leq n$ for each $j \in E$, $\sup_{j\in E} dim_{\mathbf{L}} X_j = n$.

B.1.9. Notes. Conditions $2.8.(2(iii), (5(ii,iii))$ and restrictions on f_i^j in $8.(8(a))$ are additional in comparison with the classical case. They are imposed in § 8, since there exists a continuous non-linear retraction $r : s_j \to \partial s_j$ for the non-Archimedean field **L**, which may be constructed with the help of p-subdivision. If f is simplicial on each polyhedron M and $2.8.(2(iii))$ is accomplished, then $dim_{\mathbf{L}} g(M) = dim_{\mathbf{L}} f(M)$.

Here we mean by ANRU an ultra-uniform space X such that under embedding into an ultra-uniform space Y it is the uniformly continuous retract $r : V \to X$ of its uniform neighborhood V, $X \subset V \subset Y$. We denote by $U(X,Y)$ for two ultra-uniform spaces X and Y an ultra-uniform space of uniformly continuous mappings $f : X \to Y$ with the uniformity generated by a base of the form $W = \{(f,g) | (f(x), g(x)) \in V$ for each $x \in X\}$, where $V \in \mathsf{V}$, V is a uniformity on Y corresponding to $\mathbf{P}_Y$.

Here we call an ultra-uniform space Y injective, if for each ultra-uniform space X with $H \subset X$ and an uniformly continuous mapping $f : H \to Y$ there exists an uniformly continuous extension $f : X \to Y$.

B.1.10. Theorem. *$B(c_0(\mathbf{L},A),0,1)$ is an injective space for $card(A) < \aleph_0$.*

Proof may be done analogously to theorem 9 on p. 40 in [Isb64].

B.1.11. Theorem. *Each uniform polyhedron P over $\mathbf{L}$ is ANRU.*

Proof. Since $a = \inf_{i \neq j \in F} dist(s_i, s_j) > 0$, $b = \sup_{i \in F} diam(s_i) < \infty$, then there exists a uniform covering U such that for each s_i there exists a/p-clopen neighborhood $V_i \in U$. Each s_i is an uniform retract V_i, $r_i : V_i \to s_i$, consequently, there exists uniformly continuous retraction $r : S = \bigcup_{V_i \in U} V_i \to P$ such that $r|_{s_i} = r_i$ for each i, since $\sup_{i \in F} diam(s_i) < \infty$. The neighborhood S is uniform, since $St(M,U) = S$.

B.1.12. Note. Further for uniform spaces are considered uniformly continuous mappings and uniform polyhedra and for topological spaces continuous mappings and polyhedra if it is not specially outlined.

B.1.13. Lemma. *Let $(X,\mathbf{P})$ be an ultra-uniform strongly zero-dimensional space, P be a polyhedra over $\mathbf{L}$, $A_1,\dots,A_q$ are non-intersecting closed subsets in X, $q \in \mathbf{N}$, $card(P) \geq q$. Then there exists an uniformly continuous mapping $f : X \to P$ such that $f(A_i) \cap f(A_j) = \emptyset$ for each $i \neq j$.*

Proof. There exists a disjoint clopen covering V_j for X satisfying $A_j \subset V_j$ for each $j = 1,\dots,q$ (see Theorem 6.2.4 [Eng86]). Then we can take locally constant mapping f with $f(V_j) \subset s_j \subset P$.

B.1.14. Lemma. *Each non-stretching mapping $f : E \to P$ has non-stretching continuation on $(\tilde{X},\rho)$, where P is a uniform polyhedron over $\mathbf{L}$, $E \subset X$.*

Proof. There exists an embedding $P \hookrightarrow c_0(\mathbf{L},A)$ for $card(A) = w(P)$ due to Lemma 4 [Lud95]. We choose $f : E \to c_0(\mathbf{L},A)$ with $f = (f^i : i \in A)$, $f^i : E \to \mathbf{L}e_i$, where $(e_i : i \in A)$ is the orthonormal basis in $c_0(\mathbf{L},A)$ and $\inf_{(i \neq j, s_i \text{ and } s_j \subset P)} dist(s_i, s_j) > 0$, s_i are simplexes from P.

B.1.15. Definition. An ultra-uniform space $(X,\mathbf{P})$ is called LE-space, if each uniformly continuous $f : Y \to \mathbf{L}$ has an uniformly continuous extension on all X, where $Y \subset X$.

B.1.16. Theorem. *An ultra-metric space X is an LE-space if and only if $\tilde{X} = inv-\lim_m\{X_m, f^m_n, E\}$, where X_m are fine spaces and bonding mappings $f^m_n : X_m \to X_n$ are uniformly continuous for each $m \geq n \in E$.*

Proof. Let us consider $X_A = B(c_0(\mathbf{L},A),0,1)$ for $card(A) \geq \aleph_0$ or $X_A = \mathbf{L^n}$ which are not fine spaces, since on X_A there are continuous $f : X_A \to \mathbf{L}$ which are not uniformly continuous. Then for the embedding of the non-fine space $X_A \hookrightarrow c_0(\mathbf{L},A)$ there is not compact H with $X_A \subset H \subset c_0(\mathbf{L},A)$. Therefore, in X_A there exists a countable closed subset of isolated points $Y = \{x_i : i \in \mathbf{N}\}$ and a continuous function $f : Y \to \mathbf{L}$ with $|f(x_j) - f(x_i)|/|x_j - x_i| > c_i > 0$ for each $j \geq i$ and $\lim_{i \to \infty} c_i = \infty$ such that f has a continuous extension g on X_A. Indeed, there exists a clopen neighborhood W, that is a b-enlargement (with $b > 0$) relative to the ultra-metric ρ_A on X_A, so there is a continuous retraction $r : W \to Y$, $g(x) = const \in \mathbf{L}$ on $(X_A \setminus W)$, $g(x) = f(r(x))$ on W, $g|_Y = f$. Therefore, g is continuous and uniformly continuous on X_A (see also Theorem 1.7 [CI60]).

B.2. Polyhedral Expansions

B.2.1. Theorem. *Each complete ultrauniform space* $(Y,\mathbf{P})$ *is a limit of an inverse spectra of ANRU* Y_j*, where* Y_j *are embedded into complete locally* $\mathbf{L}$*-convex spaces.*

Proof. In view of Corollary 1.5 there exists a uniform embedding $Y \hookrightarrow \prod_{\rho\in\mathbf{P}} c_0(\mathbf{L},A_\rho) =: H$. In each $c_0(\mathbf{L},A_\rho)$ may be taken the orthonormal basis $\{e_{j,\rho} : j \in A_\rho\}$, $card(A_\rho) = w(Y_\rho)$ and define canonical neighborhoods $U(f,b;(j_1,\rho_1),\dots,(j_n,\rho_n)) := \{q \in H : |\pi_{j_i,\rho_i}(q) - \pi_{j_i,\rho_i}(f)| < b$ for $i = 1,\dots,n\}$, where $\pi_{j,\rho} : H \to \mathbf{L}e_{j,\rho}$ are projectors, $f \in H$, $b > 0$, $n \in \mathbf{N}$. Each clopen subset $Z_b := H \setminus U(f,b;(j_1,\rho_1),\dots,(j_n,\rho_n))$ is uniformly continuous (non-stretching) retract of $Z_{b/p}$, that is Z_b is ANRU. Analogously each finite intersections $\bigcap_{l=1}^q Z(f_l,b_l,(j_1^l,\rho_1^l),\dots,(j_n^l,\rho_n^l)) = \bigcap_{l=1}^k Z_{l,b_l}$ are also ANRU, since $Z_{k,b_k} \supset \bigcap_{l=1}^k Z_{l,b_l}$ and a retraction $g_k : Z_{k,b_k/p} \to Z_{k,b_k}$ produces non-stretching relative to the corresponding pseudo-ultra-metric retraction $g_k : \bigcap_{l=1}^k Z_{l,b_l/p} \to \bigcap_{l=1}^k Z_{l,b_l}$. Hence $g_q \circ \dots \circ g_1 = g : \bigcap_{l=1}^q Z_{l,b_l/p} \to \bigcap_{l=1}^q Z_{l,b_l}$ is the (non-stretching) retraction. Let S be the ordered family of such $\bigcap_{l=1}^q Z_{l,b_l} \subset H \setminus Y$, then $\bigcap_{Z\in S}(H\setminus Z) = Y$. Further as in the proof of Theorem 7.1 [Isb61].

B.2.2. Lemma. *Let* (X,ρ_X) *and* (Y,ρ_Y) *be ultra-metric spaces,* $f : X \to Y$ *be a continuous mapping such that* $f|_H$ *is a b-mapping, where H is dense in X,* $b > 0$*. Then X and Y may be embedded into a Banach space W over* $\mathbf{L}$ *such that X with equivalent ultra-metric in W and* $\|f(x)-x\| \le b$ *for each* $x \in X$*, that is, f is a b-shift or a b-mapping.*

Proof. From Theorem 1.4 and Lemma 1.3 and Corollary 1.6 it follows that there exists such embedding of X into the corresponding $W = c_0(\mathbf{L},A)$ with $card(A) = w(X)$ with the disjoint clopen covering $V = \{B(c_0(\mathbf{L},A),x_j,b_j) =: B_j : x_j \in H, \infty > b_j > 0, j \in F\}$. Let $Y_j := f(B_j) \subset Y$, consequently, $diam(Y_j) \le b$, since f is the b-mapping, that is f realizes the covering of X consisting from subsets of diameters not greater than b. Then $\rho_X(x',x") \le \max(\rho(x',x_j),\rho(x_j,x"))$, where $x" \in f^{-1}(y)$, $y \in Y_j$, $x' \in B(c_0(\mathbf{L},A),x_j,b_j)$, $f(x') = y$, $diam(f^{-1}(y)) \le b$. Let $x_i \ne x_j$, this is equivalent to $B_i \cap B_j = \emptyset$ and is equivalent to $\rho_X(x_i,x_j) > b$, consequently, $Y_i \cap Y_j = \emptyset$ and $Y = \bigcup_{j\in F} Y_j$. In view of continuity of f there exists the embedding of Y_j into B_j, since f is the b-mapping, $w(Y) \le w(X)$, $0 < b_j \le b$ and due to theorem 2.4.

B.2.3. Lemma. *Let there exists a non-stretching* (*uniformly continuous*) *mapping* $f : R \to P$ *and a non-stretching* (*uniformly continuous*) *permissible modification* $g : M \to P$*, where R is a complete ultra-metric space* (*LE-space respectively*)*, P is a polyhedron, M is a subspace in R; if R is a polyhedron let M be a sub-polyhedron. Then g has a non-stretching* (*uniformly continuous respectively*) *extension on the entire R and this extension is a permissible modification of f.*

Proof. For a complete ultra-metric space R the mapping g has the non-stretching (uniformly continuous) extension on the completion of M which coincides with the closure $\bar{M}$ of M in R, since R and P are complete. The space P is complete, since it is ANRU by Theorem 1.11 (see also Theorem 1.7 [Isb59], Theorems 8.3.6 and 8.3.10 [Eng86]). For the embedding $R \hookrightarrow c_0(\mathbf{L},A)$ with $card(A) = w(R)$ by Theorem 1.4 it may be taken due to Corollary 1.6 the disjoint clopen covering V such that each $W \in V$ has the form $W = R \cap B(c_0(\mathbf{L},A),x,r_W)$, where $r_W > 0$, $x \in R$, $\sup_{W\in V} r_W \le p^{-j}$ (for each $j \in \mathbf{N}$ there exists such V).

In view of uniform continuity of f and uniformity of the polyhedron P there exists V

such that for each W from V there exists a simplex $T \subset P$ with $f(W) \subset T$. The space $c_0(\mathbf{L},A)$ has the orthonormal basis $\{e_j : j \in A\}$. If f is linear on no any $[a,b] \subset R$, then for the construction of g may be used lemma 2.14 or Theorem 1.16. This may be done with the help of transfinite induction by the cardinality of sets of vertices of simplices from P (or using the Teichmüller-Tukey lemma) and with the modification of the proof of Lemmas 1 in [Fre37, Isb61]. In general let $g(\bar{M})$ contains each zero-dimensional simplex $T^0 \subset P$. If it is not so, then there exists a point $b = f^{-1}(T^0)$ in which g is not defined. If f is non-linear on each $[a,b] \subset R$ with $a \in \bar{M}$, then $g(b) = f(b)$. If f is linear on such $[a,b]$, then from $\rho(a,b) < \infty$ it follows that $[a,b]$ is homeomorphic with the compact ball $B(\mathbf{L},a,\rho(b,a))$ in $\mathbf{L}$ and $f([a,b])$ is homeomorphic with $B(\mathbf{L}, f(a), \|f(b) - f(a)\|)$. Then in $\bar{M} \cap [a,b] = Y$ there exists y such that $\rho(y,a) = \sup_{x \in Y} \rho(a,x) = t$. The subspace Y is covered by the finite number of $W \in V$. For $y \neq a$, that is $t > 0$, $f([a,b])$ is compact and is contained in the finite number of simplexes from P, consequently, there exists the permissible modification of g on $[a,b]$ also.

Let E and F be two subsets in R. We denote by $sp(E,F,f)$ the subspace $cl((\bigcup_{(a\in E, b\in F, f|_{[a,b]} \text{ is } \mathbf{L}-\text{ linear })}[a,b]) \cup E \cup F)$ in R, where $cl(S)$ denotes the closure of S in R for $S \subset R$. If $B = \{q : f^{-1}(T^0) = q, T^0 \subset P, g \text{ is not defined in } q\}$, then by the Teichmüller-Tukey lemma there exists the extension of g on $sp(\bar{M},B,f) =: M_0$. Let $M_j := sp(\bigcup_{i<j} M_i, B, f)$, where $B = \bigcup_{(T^j \text{ is not the subset of } g(M))} T^j$, T^j are simplexes from P with the cardinality of sets of vertices equal to j, where $j \leq w(P)$. From Lemma 2.2 it follows that conditions $1.8.(5(i-iii))$ may be satisfied on $M_j \cap R$. Considering vertices of s from $R \cap f^{-1}(T^j) \setminus \bigcup_{i<j} M_i$ we construct g on $M_j \subset R$. Since $R \hookrightarrow c_0(\mathbf{L},A)$, then $\sup_j M_j = R$, where M_j are ordered by inclusion: $M_j \supset M_i$ for each $i \leq j$.

Henceforth, we assume that for uniformly continuous mapping $f : Y \to P$ are satisfied Conditions 1.14 and 1.16, where $Y \subset (X, \rho)$.

B.2.4. Lemma. *Let $f : M \to P$ is irreducible, M and P are polyhedrons, N is a sub-polyhedron in M, Q is a sub-polyhedron in P, $f(N) \subset Q$, then a mapping $f : N \to Q$ is irreducible.*

Proof. Let $f : N \to Q$ be reducible, that is there exists an permissible modification $g : N \to Q$ with $g(N)$ not contained in $f(N)$, $q \in f(N) \setminus g(N)$. In view of Lemma 2.3 there exists the extension $g : M \to P$. Let $r > 0$ is sufficiently small and $U := N_r = \{y \in M :$ there exists $x \in N$ with $\rho(x,y) \leq r\}$ be the r-enlargement of the subspace N such that $q \notin g(N_r)$. Since M and P are (uniform) polyhedra and f is (uniformly) continuous and U is clopen in M, then there exists p^j-subdivision M' and clopen polyhedron H with $N_{r/p} \subset N \subset U \subset M'$ such that $h|_H = g|_H$, $M' \setminus H$ is the sub-polyhedron in M', $h|_{M\setminus H} = f|_{M\setminus H}$. Then $q \notin h(H)$ and from the irreducubility of f it follows that $q \notin f(M \setminus H)$, consequently, h is the permissible modification of f and $h(M)$ is not the subset of $f(M)$. This contradiction lead to the statement of this lemma.

B.2.5. Lemma. *Let $f : M \to P$, M and P be polyhedrons over* $\mathbf{L}$. *Then the condition of irreducibility of f is equivalent to that each sub-polyhedron in $Q \subset P$ is essentially covered.*

Proof. If f is irreducible, then due to Lemma 2.4 each sub-polyhedron Q is essentially covered. Let vice versa each Q is essentially covered and f has the permissible modification g with $P = f(m)$ not contained in $g(M)$. From $f(\partial M) = g(\partial M)$ and that f is essential the contradiction with $\{P$ is not contained in $g(M)\}$ follows, consequently, f is irreducible.

B.2.6. Lemma. *Let P and M be polyhedrons. If $f : M \to P$ has only admissible modifi-*

cations, then f is irreducible.

Proof. From $f(\partial M) = g(\partial M)$ and that g is the permissible modification of f it follows that each sub-polyhedron Q from P is essentially covered due to lemma 3.3. In view of Lemma 2.5 f is irreducible.

B.2.7. Lemma. *Let there is an inverse spectra $S = \{R_m, f^m_n, E\}$ of polyhedra R_m over* $\mathbf{L}$*, f^m_n are simplicial mappings, $g^l : R_l \to P$, $g^n = g^l \circ f^n_l$ for a fixed l, $f_n = inv - \lim_m f^m_n$, $g = g^l \circ f_l$, $R = inv - \lim S$, $f_n : R \to R_n$, E is linearly ordered. If $g : R \to P$ is reducible for a polyhedron P, then for almost all n (that is, there exists $k \in E$ such that for each $n \geq k$) $g^n : R_n \to P$ are reducible.*

B.2.8. Lemma. *If $f : M \to N$, $g : N \to T$, $f(M) = N$, where M and N are polyhedra, then from g is inessential (reducible) it follows $f \circ g$ is inessential (reducible).*

B.2.9. Lemma. *Suppose that there is given an irreducible inverse mapping system $S = \{P_m, f^m_n, E\}$ of polyhedra P_m over* $\mathbf{L}$*, M is closed in $P = inv - \lim S$ such that $M_n := f_n(M)$ are sub-polyhedra in P_n and for each $m > l$ permissible modifications g^m_l for f^m_l are given (on the entire P_m) and for each $n > m$ mappings $g^m_l \circ f^n_m$ are permissible modifications of f^m_l. Then the inverse mapping system $S_M := \{M_n, g^m_l \circ f^n_m, E_l\}$ is irreducible, where $E_l = \{n \in E : n \geq l\}$.*

Proof is analogous to the proofs of Lemmas IV.30.V-VII in [Fre37] using the preceding lemmas (see also [Isb61]). Indeed, in Lemma 2.8, if f is $\mathbf{L}$-linear on $[a, b] \subset M$ and g is $\mathbf{L}$-linear on $[f(a), f(b)]$, then $f \circ g$ is $\mathbf{L}$-linear on $[a, b]$. In Lemma 2.9 from surjectivity and irreducibility of f^m_l it follows surjectivity and irreducibility of $g^m_l \circ f^n_m$ due to Lemmas 2.4, 2.5 and 2.8. Since E_l is cof-inal with E, then for each W open in M there exists V open in P such that $W = M \cap V$.

B.2.10. Lemma. *If T is a simplex from a p^j-subdivision P' of polyhedron P then for each clopen neighborhood $U \supset T$ such that U is a sub-polyhedron in P' there exists a mapping $k : P \to P$ such that $k|_U$ is simplicial and $k(U) = T$.*

Proof. In view of Theorem 1.11 there exists the retraction $r : P \to U$ (it is uniform if P is the uniform polyhedron), then analogously to the proof of Lemma VIII [Fre37].

B.2.11. Lemma. *Let P be a polyhedron, $f : M \to P$ and $g : M \to P$ are two b-close mappings (that is, $\rho(f(a), f(b)) \leq b$ for each $a \in M$), then there exists k from lemma* 3.10 *such that kg and f are b-close and kg is a permissible modification of f.*

B.2.12. Lemma. *Let $f : P \to Q$ be a non-stretching mapping of a uniform polyhedron P onto a uniform polyhedron Q over* $\mathbf{L}$*. Then there exists a p^j-subdivision P' of P and a normal mapping $g : P \to Q$ such that g is a permissible modification of f.*

Proof. For each $b > 0$ in view of uniform continuity of f there exists p^j-subdivision P' of P and p^i-subdivision of Q and a simplicial mapping $h : P' \to Q$, which is b-close to f. Indeed, simplexes $T^l \subset P'$ are disjoint and clopen for them due to Lemmas 3 and 4 [Lud95] h can be chosen such that: (i) $|h(e) - f(e)| < b$ for linearly independent vertices $e_{l,j} \in T^l$, that is, $\{(e_{l,j} - e_{i,0}) : j \in A_l\}$ are linearly independent, $e_{l,0}$ is a marked vertex from T^l, $card(A_l) = dim_{\mathbf{L}} T^l$ and (ii) $h(T^l)$ are simplexes in Q with $diam(h(T^l)) \leq diam(T^l)$ for suitable $b > 0$ and p^i-subdivision Q' of Q, where $h|_{T^l}$ are affine mappings for each l. This is possible due to uniformity of polyhedra P and Q. Taking $g = k \circ h$, where k is the suitable mapping from Lemma 2.11 we get the desired g.

B.2.13. Lemma. *Let g be a permissible modification of $f : R \to P$ and $h : P \to Q$ is a normal mapping, where P and Q are uniform polyhedra. Then $h \circ g$ is the permissible* *modification of* $h \circ f$.

B.2.14. Lemma. *Let* $\{P_n, f^n_m, E\} = S$ *be an irreducible inverse system of uniform polyhedrons over* $\mathbf{L}$*,* M *be closed in* $P = inv - \lim S$*,* $M_n = f_n(M)$ *be sub-polyhedra in* P_n*,* $\{Q_k, g^k_l, F\}$ *be an inverse spectra of polyhedra* Q_k*, which appear by* $p^{j(k)}$*-subdivisions of* $P_{n(k)}$*,* g^k_l *be normal and permissible modifications of* $f^{n(k)}_{n(l)}$*, where* F *is co-final with* E*,* $N_k = g_k(M)$*,* $g^k_l|_{N_k}$ *and* $g_l|_M$ *are irreducible. Then* N_k *are polyhedra.*

Proof follows from Lemmas 3.7-3.12 (see also [Fre37, Isb61]).

B.2.15. Lemma. *Suppose that for a complete ultra-metric space* R *there is a non-stretching mapping* $f : R \to P$*,* $f(R) = P$*,* P *is a uniform polyhedron over* $\mathbf{L}$*. Then for each* $b > 0$ *there exists a b-mapping* $g : R \to Q$ *and* Q *is a uniform polyhedron over* $\mathbf{L}$ *such that for sufficiently fine* p^j*-subdivision* Q' *of a polyhedron* Q *there exists a normal mapping* $h : Q \to P$ *and* $h \circ g$ *is a non-stretching permissible modification of* f*.*

Proof. For R there exists the embedding $R \hookrightarrow c_0(\mathbf{L}, A)$ with $card(A) = w(R)$ by theorem 2.4 and a clopen neighborhood S with $R \subset S \subset R(r)$ that is a uniform polyhedron due to Corollary 1.6, where $R(r)$ denotes the $r = b/p$-enlargement of R. By Lemma 2.2 there exists the sub-polyhedron Q with $R \subset Q \subset S$ and the b'-mapping $g : R \to Q$. If $[a,z] \subset S$ and $f|_{[a,z]}$ is $\mathbf{L}$-linear, then we can choose g such that it is linear on $[a,z]$ and $g(a) \neq g(z)$ when $f(a) \neq f(z)$. From the completeness of R and lemma 2.14 it follows that f has the non-stretching extension $f : S \to P$. Then there are g, Q and non-stretching $h : Q \to P$ for sufficiently small $r > 0$ and $b' > 0$. For h due to Lemma 2.12 there exists the permissible modification and normal $k : Q' \to P$ such that $h \circ g(R) = P$, $f(\partial R) = h \circ g(\partial R)$ for $\partial R \neq 0$, that is, when R contains simplexes T with $dim_{\mathbf{L}} T > 0$. Such k, h and g may be constructed on each simplex T from Q' and then on the entire space.

B.2.16. Lemma. *Suppose that* R *is a complete ultra-metric space,* $f_n : R \to P_n$ *are non-stretching* b_n*-mappings,* P_n *are uniform polyhedra over* $\mathbf{L}$*,* $n \in E$*,* E *is an ordered set such that for each* $b > 0$ *there exists* $l \in E$ *with* $0 < b_n < b$ *for* $n > l$*. Then there exists a normal irreducible inverse mapping system* $S = \{Q_m, g^n_m, F\}$*,* F *is co-final with* E*,* $inv - \lim S = R$*,* Q_m *are sub-polyhedra of* $p^{j(m)}$*-subdivisions of* $P_{n(m)}$ *and* g_m *are permissible modifications of* $f_{n(m)}$*, where* $g_n = inv - \lim_n g^n_m : R \to Q_m$*.*

Proof. In view of Lemmas 2.5, 2.6, 2.13 and 2.14 it is sufficient at first to construct S with non-stretching normal normal and surjective mappings g^m_n. This can be done due Lemma 2.15 with $g^m_l \circ f_{n(m)}$ being non-stretching permissible modifications of $f_{n(l)}$ and $inv - \lim_m g^m_l \circ f_{n(m)}$ are non-stretching permissible modifications of $f_{n(l)}$.

B.2.17. Lemma. *If the ultra-metric space* (X, ρ) *is isomorphic with* $inv - \lim\{(X_m, \rho_m); f^m_n; F\}$ *and the following conditions are satisfied:*

(1) *for each* m *there are embeddings* $q_m : X_m \hookrightarrow (E, \rho)$ *into a complete space* (E, ρ)*;*

(2) $f_m : X \to X_m$ *are projections;*

(3) (X_m, ρ_m) *are ultra-metric spaces;*

(4) *there is given a family* $\{b_m > 0 : m \in F\}$*,* $b_m \in \rho(X, X)$ *and for each* $b > 0$ *the set* $\{m : b_m > b\}$ *is finite,* $t_m := \inf_{\rho(x,y) > b_m} \rho(q_m(x), q_m(y))$*,* $\lim_m t_m = 0$*, for each* $m > n$ *and* x *and the inequality* $\rho(q_m(x), q_n \circ f^m_n(x)) < t_n$ *is accomplished. Then the mappings* $q_m \circ f_m$ *converge uniformly to the embedding* $X \hookrightarrow E$*.*

Proof. In view of Lemma 1.2 we may suppose that $\rho(X, X)$ and $\rho_m(X_m, X_m) \subset \tilde{\Gamma}(\mathbf{Q_p})$. If x and $y \in X$ and $\rho(x, y) > b_n$, then $\rho(q_n \circ f_n(x), q_n \circ f_n(y)) \geq t_n$. From conditions $(2,4,5)$

it follows the existence of k such that $\rho(q_m \circ f_m(x), q_m \circ f_m(y)) \geq t_n$ for each $m > k$ and ultra-metric ρ, consequently, $q = inv - \lim_m q_m$ is the embedding, since $\rho(q(x), q(y)) \geq t_n$.

B.2.18. Theorem. *Let $(X, \mathbf{P})$ be a complete ultra-metric space and $\mathbf{L}$ be a locally compact field, $\mathbf{L} \supset \mathbf{Q_p}$. Then there exists an irreducible normal expansion of $(X, \mathbf{P})$ into the limit of the inverse mapping system $S = \{P_n, f^n_m, E\}$ of uniform polyhedra P_n over $\mathbf{L}$, moreover, $inv - \lim S$ is isomorphic with $(X, \mathbf{P})$, in particular for ultra-metric space (X, ρ) the system S is the inverse sequence.*

Proof. From Corollary 1.5 and Theorem 2.1 it follows the existence of the expansion of $(X, \mathbf{P})$ into the uniformly isomorphic limit of the inverse mapping system $R = \{Y_j, f^j_i, F\}$ of ANRU Y_j with non-stretching f^j_i, where Y_j are the complete ultra-metric spaces. Each Y_j is closed in the finite products of the spaces $c_0(\mathbf{L}, A_k)$. From Lemmas 2.2-2.17 it follows the existence of the irreducible normal uniform polyhedral expansion for each Y_j, moreover, using the permissible modifications of g^j_i of f^j_i and the same lemmas we can construct the system of the entire space $(X, \mathbf{P})$ with the same properties.

We consider further uniform coverings V corresponding to the uniform polyhedra $P = \bigcup_{W \in V} W$, which due to theorem 2.11 are ANRU. Let ρ be the ultra-metric in X and $\rho(X, X) \subset \tilde{\Gamma}(\mathbf{L})$. We can associate with V a p^k-nerve with $k \in \mathbf{Z}$, that is, an abstract simplicial complex N_k vertices of which are elements from V. Its simplexes are the spans of (pulled on) vertices W_j satisfying $\rho(W_j, W_i) \leq p^k b$, where $b = \sup_{W \in V} diam(W) < \infty$, each rib $[W_j, W_i]$ from s has the lenghts no less than $t = \inf_{W_j \neq W_i \in V} \rho(W_j, W_i) > 0$. Then from $N_k \hookrightarrow c_0(\mathbf{L}, A_k)$ with $card(A_k) = w(N_k)$ it follows that each s is uniformly isomorphic with some ball $B(c_0(\mathbf{L}, A), 0, 1)$, where $card(A) = m \leq w(N_k)$. With each V is associated the equivalence relation: xRy if and only if there exists $W \in V$ such that x and $y \in W$. Since V is disjoint and clopen, then the quotient mapping $f : X \to X/R$ is defined. With each V is associated the partition of the unity $\{f_W : W \in V\}$, $f_W : X \to B(\mathbf{L}, 0, 1)$, $f_W(x) = 1$ for $x \in W$ and $f_W(x) = 0$ for $x \notin W$, $\{f_W : W \in V\}$ is subordinated to V. There are the canonical non-stretching mappings $F_k : X \to N_k$. If X is compact, then X/R is the finite discrete space and $dim_{\mathbf{L}} N_k = n \in \mathbf{N}$. Into each V may be refined a disjoint clopen uniform covering K with $\sup_{W \in K} diam(W) \leq bp^{-j}$, where $j \in \mathbf{N}$. That is, V has the uniform strict shrinking.

We can consider the sequence of such shrinkings: $V^{m+1} \subset V^m$ with $b_m = bp^{-m}$, where $m \in \mathbf{N}$. With each V^m is associated $p^{k(m)}$-nerve $N_{k(m)}$. Let $k(m) \geq -m$, $k(m+1) \leq k(m)$ for each $m \in \mathbf{N}$ and $\lim_{m \to \infty} k(m) = -\infty$. If x is an isolated point in X, then there exists $n \in \mathbf{N}$ with $\max(b_n, p^{k(n)} b_n) < \inf_{y \in X \setminus \{x\}} \rho(x, y)$. Then the simplex $s \subset N_{k(m)}$ with $x \in s$ and $m \geq n$ is zero-dimensional over $\mathbf{L}$, that is, $s = \{x\}$.

By the construction of N_k for each simplex $s_{m+1} \subset N_{k(m+1)}$ there exists $s_m \subset N_{k(m)}$ with $f^{m+1}_m(s_{m+1}) \subset s_m$, where f^j_i are the bonding mappings of the inverse sequence $S = \{N_{k(m)}, f^m_i, \mathbf{n})$. Each f^{m+1}_m is non-stretching, since decreases the distance at least into p times and $b_m / b_{m+1} \geq p$. If $x \neq y$, then there exists n with $\max(b_n, b_n p^{k(n)}) < \rho(x, y)$, consequently, for each $m > n$ there are disjoint simplexes s and $s' \subset N_{k(m)}$ with $x \in s$ and $y \in s'$. Therefore, there exists the uniformly continuous mapping $g : X \to inv - \lim S$, where $g(x) = inv - \lim_m \{s_m, f^m_i\}$ and $s_m \ni x$ for each $m \in \mathbf{N}$. Therefore, the uniformly continuous projectors $f_m : X \to N_{k(m)}$ are defined, since for each $b > 0$ there exists $r \in \mathbf{N}$ such that $b_m p^{k(m+r)-k(m)} < b$ and $f_m(W) = f^{m+r}_m \circ f_{m+r}(W)$ and $diam(f_m(W)) < b$, where $W \in V^{m+r}$, $f_{m+r}(W)$ belongs to clopen star of the corresponding vertex $v \in N_{k(m+r)}$. Further as in

Lemma IV.33 in [Isb64] we can verify that $g(X) = inv - \lim S$ and g is the uniform isomorphism.

Evidently, $dim_{\mathbf{L}} N_{k(m)}$ may be from 0 for $k(m) = -m$ up to $card(A)$ with $card(A) = w(X)$. For $k(m) > -m$ in the inverse sequence S the mappings f_m^{m+1} may map simplexes s from $N_{k(m+1)}$ into simplexes q from $N_{k(m)}$ of lower dimension over $\mathbf{L}$, for example, when $W_{m+1} \subset W_m$, $W_m \in V^m$, $W_m = B(c_0(\mathbf{L}, A_j), x, r)$, $W_{m+1} = B(c_0(\mathbf{L}, A_n), x', r/p)$, $card(A_n) > card(X/R_m) \ge dim_{\mathbf{L}} N_{k(m)} \ge card(A_j)$, since $dim_{\mathbf{L}} N_{k(m+1)} \ge card(A_n)$ for $k(m+1) > -m - 1$.

For the complete ultra-uniform space $(X, \mathbf{P})$ we can consider the base of uniform coverings $\{V_\rho^n : n \in \mathbf{N}, \rho \in \mathbf{P})$, where each V_ρ^n is given relative to the considered ρ, $S = \{N_{\rho,k(m)}, f_{\rho',k(m')}^{\rho,k(m)}, \mathbf{P} \times \mathbf{N}\}$. To each V_ρ^m there corresponds $N_{\rho,k(m)}$; $\rho' \le \rho$ if and only if $\rho'(x,y) \le \rho(x,y)$ for each x and $y \in X$; $(\rho', m') \le (\rho, m)$ if and only if $\rho' \le \rho$ and $m' \le m$.

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